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Common fixed points using (ψ,ϕ) - type contractive maps in fuzzy metric spaces

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Abstract: This In this paper, we define new control functions to give unique fixed point in fuzzy metric space. A fruitful contractive condition of (ψ, ϕ)- type is used to obtain common fixed point theorem for two maps in fuzzy metric spaces. We extend the existing results in metric space to fuzzy metric space using these control functions. The first theorem is the extension of the result of Zhang and Song (2009) under the required contractive conditions. Second result is analogous to the result of Doric (2009) in metric spaces.

Keywords: ψ-contractive mappings, fuzzy metric space, control functions, weak contraction

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1. Introduction and preliminaries

Zadeh (1965) investigated fuzzy set theory. Many authors utilized the concept of fuzzy set theory in metric space in number of ways. Banach contraction principal is the elate result of fixed point theory. Several authors have developed different contractive conditions to find fixed point in metric space (Dutta & Choudhury, 2008; Gupta & Mani, 2014; Gupta, Mani & Tripathi, 2012; Gupta, Saini, Mani & Tripathi, 2015; Song, 2007; Song & Xu, 2007). Kramosil and Michalek (1975) defined fuzzy metric space using the concept of $t-$norm. George and Veeramani (1994) modified the notion of fuzzy metric spaces by using continuous $t-$norm. Gregori and Sapena (2002) also explored the Banach contraction principal to fuzzy contractive mapping on complete fuzzy metric space. Particularly, Mihet (2008) introduced the concept of fuzzy contractive mappings, which is one of the weak contractions in fuzzy metric space. In 1997, the concept of weak contraction was defined (Alber & Guerre-Delabriere, 1997) for single valued maps on Hilbert spaces.

Rhoades (2001) introduced weakly contractive mapping in metric space by defining a map $T: X \to X$, which satisfy the condition $d(Tx,Ty) \leq d(x,y) - \phi(d(x,y))$, where $x, y \in X$ and $\phi: [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function such that $\phi(t) = 0$ if and only if $t = 0$.

Zhang and Song (2009) proved a unique common fixedpoint theorem of hybrid generalized ϕ-weak contraction for two maps T, S on complete metric space X. The result is given below.

Theorem 1.1 (Zhang & Song, 2009) Let (X,d) be a complete metric space and $T, S: X \to X$ be two mappings such that for all $x, y \in X$,

$$d(Tx,Sy) \leq M(x,y) - \phi(M(x,y)), \tag{1}$$

where $\phi: [0,\infty) \to [0,\infty)$ is a lower semi-continuous function with $\phi(t) > 0$ for $t \in (0,\infty)$ and
$\phi(0) = 0$,

$$M(x,y) = max\left\{d(x,y), d(Tx,x), d(Sy,y), \tfrac{1}{2}[d(y,Tx) + d(x,Sy)]\right\} \tag{2}$$

Then there exists a unique $u \in X$ such that $u = Tu = Su$. Doric (2009) established a fixed point theorem which generalized the result of Zhang and Song (2009) using control functions, which is given below;

Theorem 1.2 (Doric, 2009) Let (X,d) be a complete metric space and $T, S: X \to X$ be two mappings such that for all $x, y \in X$,

$$\psi(d(Tx,Sy)) \leq \psi(M(x,y)) - \phi(M(x,y)), \tag{3}$$

where

1. $\psi: [0,\infty) \to [0,\infty)$ is a continuous monotone non-decreasing function with $\psi(t) = 0$ if and only if $t = 0$,
2. $\phi: [0,\infty) \to [0,\infty)$ is a lower semi-continuous function with $\phi(t) = 0$ if and only if $t = 0$, and

$$M(x,y) = max\left\{d(x,y), d(Tx,x), d(Sy,y), \tfrac{1}{2}[d(y,Tx) + d(x,Sy)]\right\} \tag{4}$$

Then there exists a unique $u \in X$ such that $u = Tu = Su$. The aim of our work is to prove above results in fuzzy metric spaces. The first theorem is the extension of the result of Zhang and Song (2009) under the different contractive conditions using control functions. Second result is analogous to the result of Doric (2009) in metric spaces.

Definition 1.1 (Schweizer & Sklar, 1960) A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is continuous $t-$ norm if $*$ satisfies the following conditions

(T-1) $*$ is commutative and associative;

(T-2) $*$ is continuous;

(T-3) $a * 1 = a$ for all $a \in [0,1]$;

(T-4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Definition 1.2 (George & Veeramani, 1994) The 3-tuple $(X, M, *)$ is said to be fuzzy metric space if X is an arbitrary set, $*$ is continuous t -norm and M is fuzzy set on $X^2 \times [0,\infty)$ satisfying the following conditions for all $x, y, z \in X$ and $s, t > 0$,

(FM-1) $M(x,y,t)>0$
(FM-2) $M(x,y,t) = 1,\ \forall\, t > 0$ iff $x = y$;
(FM-3) $M(x,y,t) = M(y,x,t)$;
(FM-4) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$;
(FM-5) $M(x,y,.): [0,\infty) \to [0,1]$ is continuous.

The triplet $M(x,y,t)$ can be taken as the degree of nearness between x and y with respect to $t \geq 0$.

Remark 1.1 (Shen, Qiu & Chen, 2013) Since $*$ is continuous, it follows from (FM-4) that the limit of the sequence in fuzzy metric space is uniquely determined.

Let $(X, M, *)$ is a fuzzy metric space then the following condition also holds:

(FM-6) $lim_{t\to\infty} M(x,y,t) = 1.$

Lemma 1.1 (Grabiec, 1988) In fuzzy metric space $(X, M, *)$, $M(x,y,\cdot)$ is non-decreasing for all $x, y \in X$.

Definition 1.3 (George & Veeramani, 1994) Let $(X, M,*)$ be a fuzzy metric space. Then a sequence $\{x_n\} \in X$ is said to be

a) Convergent to a point $x \in X$ if for all $t > 0$, $lim_{n\to\infty} M(x_n, x, t) = 1$;
b) Cauchy sequence if for all $t > 0$ and $p > 0$, $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$;
c) A fuzzy metric space $(X, M,*)$ is said to be complete if and only if every Cauchy sequence in X is convergent.

2. Main results

Theorem 2.1 Let $(X, M,*)$ be a complete fuzzy metric space and $T, S: X \to X$ be two mappings such that for all $x, y \in X$,

$$M(Tx, Sy, t) \geq N(x, y, t) + \phi(N(x, y, t)), \tag{5}$$

where $\phi: [0,1] \to [0,1]$ is a upper semi-continuous function such that $\phi(t) > 0$ for $t \in (0,1)$ and $\phi(1) = 0$, and

$$N(x, y, t) = \min\{M(x, y, t), M(Tx, x, t), M(Sy, y, t), M(y, Tx, t) * M(x, Sy, t)\} \tag{6}$$

then there exists a unique $u \in X$ such that $u = Tu = Su$.

Proof. To prove our result, we follow the following steps.

Step- I We show that $N(x, y, t) = 1$ if and only if $x = y$, is a common fixed point of T and S.

Infact, if $x = y = Tx = Ty = Sx = Sy$, then $N(x, y, t) = 1$. Let $N(x, y, t) = 1$, then
$M(x, y, t) \geq N(x, y, t), M(Tx, x, t) \geq N(x, y, t), M(Sy, y, t) \geq N(x, y, t)$
we have $x = y = Tx = Ty = Sx = Sy$.

Step- II Let x_0 be a given point in X. We construct the sequence $\{x_n\}$ for $n \geq 0$ inductively as follows.

Choose a sequence $\{x_n\} \in X$ so that $x_{2n+1} = Sx_{2n}$, $x_{2n+2} = Tx_{2n+1}$ and prove that $M(x_{n+1}, x_n, t) \to 1$ as $n \to \infty$.

Suppose that n is an odd number. Substituting $x = x_n$ and $y = x_{n-1}$ in (5), we obtain
$M(x_{n+1}, x_n, t) = M(Tx_n, Sx_{n-1}, t)$
$\geq N(x_n, x_{n-1}, t) + \phi(N(x_n, x_{n-1}, t))$
$\geq min\{M(x_n, x_{n-1}, t), M(Tx_n, x_n, t), M(Sx_{n-1}, x_{n-1}, t), M(x_{n-1}, Tx_n, t) * M(x_n, Tx_{n-1}, t)\}$
$min\{M(x_n, x_{n-1}, t), M(x_{n+1}, x_n, t), M(x_n, x_{n-1}, t), M(x_{n-1}, x_n, t) * M(x_n, x_{n-1}, t)\}$,
which implies that $M(x_{n+1}, x_n, t) \geq M(x_n, x_{n-1}, t)$.

If we take $M(x_{n+1}, x_n, t) < M(x_n, x_{n-1}, t)$,
Then $N(x_n, x_{n-1}, t) = M(x_{n+1}, x_n, t)$ and moreover
$M(x_{n+1}, x_n, t) \geq M(x_{n+1}, x_n, t) + \phi(M(x_{n+1}, x_n, t))$,
this is a contradiction. Therefore we have

$$M(x_{n+1}, x_n, t) \geq N(x_n, x_{n-1}, t) \geq M(x_n, x_{n-1}, t) \tag{7}$$

Similarly, we can also obtain inequalities (7) in case when n is an even number.

So, the sequence $\{M(x_{n+1}, x_n, t)\}$ is non-decreasing sequence and bounded above. So there exists $r > 0$, such that $\lim_{n\to\infty} M(x_{n+1}, x_n, t) = \lim_{n\to\infty} M(x_n, x_{n-1}, t) = r$, then upper continuity of ϕ implies that $\phi(r) \leq \limsup_{n\to\infty} \phi(M(x_n, x_{n-1}, t))$.

We claim that $r = 1$. In fact, taking upper limit as $n \to \infty$ on either side of the following inequality, we have
$M(x_{n+1}, x_n, t) \geq N(x_n, x_{n-1}, t) + \phi(N(x_n, x_{n-1}, t))$,
$r \geq r + \limsup_{n\to\infty} \phi(M(x_n, x_{n-1}, t))$,
$r \geq r + \phi(r)$.
this is a contradiction unless $\phi(r) = 0$ at $r = 1$.

$$\text{Hence } \lim_{n\to\infty} M(x_{n+1}, x_n, t) = 1. \tag{8}$$

Step -III Next we prove that $\{x_n\}$ is a Cauchy sequence. To prove this, it is sufficient to prove that the sub-sequence $\{x_{2n}\}$ of $\{x_n\}$ is a Cauchy sequence. Suppose opposite that the sequence $\{x_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ such that $n(k)$ is the smallest index for which $n(k) > m(k) > k$, we have $M(x_{2m(k)}, x_{2n(k)}, t) \leq \epsilon$.

Therefore,
$M(x_{2m(k)}, x_{2n(k)}, t) = M(Tx_{2m-1}, Sx_{2n-1}, t)$
$\geq N(x_{2m-1}, x_{2n-1}, t) + \phi(N(x_{2m-1}, x_{2n-1}, t)$,
this gives $\epsilon \geq \epsilon + \phi(\epsilon)$,
this gives a contradiction with $\epsilon > 0$.

Thus $\{x_{2n}\}$ is a Cauchy equence and hence $\{x_n\}$ is a Cauchy sequence.

In complete fuzzy metric space $(X, M,*)$, there exists $u \in X$ such that sequence $x_n \to u$ as $n \to \infty$.

Step - IV Now we prove that u is a fixed point of T and S. For this suppose that $u \neq Tu$, then for $M(u, Tu, t) < 1$, there exist $N_1 \in N$ such that for any $n \geq N_1$

$M(x_{2n+1}, u, t) > M(u, Tu, t)$,

$M(x_{2n}, u, t) > M(u, Tu, t)$,

$M(x_{2n}, x_{2n+1}, t) > M(u, Tu, t)$.

Under this consideration, we have

$M(u, Tu, t) \geq N(u, x_n, t) =$
$min\{M(u, x_{2n}, t), M(Tu, u, t), M(Sx_{2n}, x_{2n}, t), M(x_{2n}, Tu, t) * M(u, Sx_{2n}, t)\} =$
$min\{M(u, x_{2n}, t), M(Tu, u, t), M(x_{2n+1}, x_{2n}, t), M(x_{2n}, Tu, t) * M(u, x_{2n+1}, t)\}$.

Letting $n \to \infty$, we have
$M(u,Tu,t)>min\{M(u,u,t),M(Tu,u,t), M(u,u,t), M(u,Tu,t)*1\}=M(u,Tu,t)$,

That is

$$N(u, x_{2n}, t) = M(u, Tu, t) \text{ as } n \to \infty. \qquad (9)$$

Since
$M(Tu, x_{2n+1}, t) = M(Tu, Sx_{2n}, t)$
$\geq N(u, x_{2n}, t) + \phi(N(u, x_{2n}, t))$,
then letting $n \to \infty$, we have
$M(Tu, u, t) \geq M(Tu, u, t) + \phi(M(Tu, u, t))$,
we obtain a contradiction. Hence $Tu = u$.

Also, $M(u, Su, t) = M(Tu, Su, t) \geq N(u, u, t)$
$+\phi(N(u, u, t)) = M(u, u, t) + \phi(M(u, u, t))$
implies $Su = u$

Suppose there exists another fixed point $v \in X$ such that $v = Tv = Sv$, then using an argument similar to the above, we get
$M(u, v, t) = M(Tu, Sv, t)$
$\geq N(u, v, t) + \phi(N(u, v, t))$
$\geq M(u, v, t) + \phi(M(u, v, t))$.

Hence $u = v$. The proof is completed.

Corollary 2.1 Let $(X, M,*)$ be a complete fuzzy metric space and $T: X \to X$ be mapping such that for all $x, y \in X$,
$M(Tx, Ty, t) \geq N(x, y, t) + \phi(N(x, y, t))$,
where $\phi: [0,1] \to [0,1]$ is a upper semi-continuous function such that $\phi(t) > 0$ for $t \in (0,1)$ and $\phi(1) = 0$, and

$$N(x,y,t)= min\{M(x,y,t), M(Tx,x,t), M(Ty,y,t), M(y,Tx,t)*M(x,Ty,t)\} \qquad (10)$$

then there exists a unique $u \in X$ such that $u = Tu$.

Example 2.1 Let $(X, M,*)$ be a complete fuzzy metric space with metric $d(x, y) = |x - y|$ and $X = [0,1]$. Let

$$Tx = \frac{x}{2} \quad and \quad Sx = \frac{x}{16}$$

for each $x \in [0,1]$. Then

$$N(x, y, t) = min\left\{\frac{t}{t + |x - y|}, \frac{t}{t + \frac{x}{2}}, \frac{t}{t + y}, \frac{t}{t + |x - y|} * \frac{t}{t + |x|}\right\},$$

$$= \begin{cases} \dfrac{t}{t + |x - y|} & \dfrac{x}{2} \leq y \leq x \\ \dfrac{t}{t + y} & y \geq x. \end{cases}$$

$$= min\left\{\frac{t}{t + |x - y|}, \frac{t}{t + \left|\frac{x}{2} - x\right|}, \frac{t}{t + \left|\frac{y}{16} - y\right|}, \frac{t}{t + \left|y - \frac{x}{2}\right|} * \frac{t}{t + \left|x - \frac{y}{16}\right|}\right\},$$

$$= \frac{t}{t + \left|y - \frac{x}{2}\right|} * \frac{t}{t + \left|x - \frac{y}{16}\right|} \quad 0 \leq y \leq x.$$

For $\phi(t) = t^2 - 1$, it is easy to show that
$M(Tx, Sy, t) \geq N(x, y, t) + \phi(N(x, y, t))$
for all $x, y \in X$. One can show that all the condition of Theorem 2.1 fulfill and T, S satisfy the Theorem 2.1.

Theorem 2.2 Let $(X, M,*)$ be a complete fuzzy metric space and $T, S: X \to X$ be two mappings such that for all $x, y \in X$,

$$\psi(M(Tx, Sy, t)) \geq \psi(N(x, y, t)) + \phi(N(x, y, t)), \qquad (11)$$

where,

1. $\psi: [0,1] \to [0,1]$ is a continuous monotone non-decreasing function with $\psi(t) = 1$ iff $t = 1$,
2. $\phi: [0,1] \to [0,1]$ is a upper semi-continuous function $\phi(t) > 0$ for $t \in (0,1)$ and $\phi(1) = 0$, and

$$N(x, y, t) = min\left\{\begin{matrix} M(x, y, t), M(Tx, x, t), M(Sy, y, t), \\ M(y, Tx, t) * M(x, Sy, t) \end{matrix}\right\},$$

then there exists a unique $u \in X$ such that $u = Tu = Su$.

Proof. For any $x_0 \in X$, we construct a sequence $\{x_n\}$ for $n \geq 0$ as $x_{2n+1} = Sx_{2n}$, $x_{2n} = Tx_{2n+1}$ and will prove that $M(x_n, x_{n-1}, t) \to 1$ as $n \to \infty$. Suppose that n is an odd number, substituting $x = x_n$ and $y = x_{n-1}$ in equation (11), we obtain
$\psi(M(x_{n+1}, x_n, t)) = \psi(M(Tx_n, Sx_{n-1}, t))$

$\geq \psi(N(x_n, x_{n-1}, t)) + \phi(N(x_n, x_{n-1}, t))$.

$\psi(M(x_{n+1}, x_n, t)) \geq \psi(N(x_n, x_{n-1}, t))$,
this implies that

$$M(x_{n+1}, x_n, t) \geq N(x_n, x_{n-1}, t). \qquad (12)$$

From triangle inequality, we have
$N(x_n, x_{n-1}, t) = min\{ M(x_n, x_{n-1}, t)$,
$M(x_{n+1}, x_n, t), M(x_n, x_{n-1}, t)$,
$M(x_{n-1}, x_{n+1}, t) * M(x_n, x_n, t)\}$
$= \left\{\begin{matrix} M(x_n, x_{n-1}, t), M(x_{n+1}, x_n, t), \\ M(x_{n-1}, x_{n+1}, t) \end{matrix}\right\}$
$\geq min\{ M(x_n, x_{n-1}, t), M(x_{n+1}, x_n, t)$,
$M(x_{n-1}, x_n, t) * M(x_n, x_{n+1}, t)\}$.

If we consider
$N(x_n, x_{n-1}, t) \geq M(x_{n+1}, x_n, t)$
$\geq N(x_n, x_{n-1}, t) \geq M(x_{n+1}, x_n, t)$,
it further implies that
$\psi\big(M(x_{n+1}, x_n, t)\big) \geq \psi\big(M(x_{n+1}, x_n, t)\big)$
$+\phi(M(x_{n+1}, x_n, t))$,

this is a contradiction. So we have

$$M(x_{n+1}, x_n, t) \geq N(x_n, x_{n-1}, t) \geq M(x_n, x_{n-1}, t)$$

this implies

$$M(x_{n+1}, x_n, t) \geq M(x_n, x_{n-1}, t). \tag{13}$$

Similarly we can prove that equation (13) is true when n is even number.

Therefore the sequence $\{M(x_{n+1}, x_n, t)\}$ is monotonically non decreasing and bounded sequence. Therefore we can write

$$\lim_{n\to\infty} M(x_{n+1}, x_n, t) = \lim_{n\to\infty} M(x_n, x_{n-1}, t) = r.$$

Letting $n \to \infty$ in the given inequality (11), we get

$$\psi(M(x_{n+1}, x_n, t)) \geq \psi(M(x_n, x_{n-1}, t)) + \phi(M(x_n, x_{n-1}, t)),$$

$$\psi(r) \geq \psi(r) + \phi(r),$$

which is contradiction, unless $r = 1$ and at $\phi(r) = 0$ Hence

$$\lim_{n\to\infty} M(x_{n+1}, x_n, t) = 1. \tag{14}$$

Next we prove that $\{x_n\}$ is a Cauchy sequence. To prove this it is sufficient to prove that sub-sequence $\{x_{2n}\}$ of $\{x_n\}$ is a Cauchy sequence. Suppose $\{x_{2n}\}$ is not a Cauchy sequence, then there existsor which we can find sub-sequence $\{x_{2m(k)}\}$ and $\{x_{2n(k)}\}$ such that $n(k)$ is the smallest index for which, $n(k) > m(k) > k$ we have

$$M(x_{2m(k)}, x_{2n(k)}, t) \leq \grave{o}. \tag{15}$$

This means that $M(x_{2m(k)}, x_{2n(k)-2}, t) > \grave{o}$. From equation (15) and the triangle inequality, we have

$$\grave{o} \geq M(x_{2m(k)}, x_{2n(k)}, t)$$
$$\geq M(x_{2m(k)}, x_{2n(k)-2}, t) * M\begin{pmatrix} x_{2n(k)-2)}, \\ x_{2n(k)-1}, t \end{pmatrix}$$
$$* M(x_{2n(k)-1}, x_{2n(k)}, t),$$
$$> \grave{o} * M(x_{2n(k)-2)}, x_{2n(k)-1}, t) * M(x_{2n(k)-1}, x_{2n(k)}, t),$$

Moreover

$$M(x_{2m(k)}, x_{2n(k)+1}, t) \geq M(x_{2n(k)}, x_{2n(k)+1}, t) * M(x_{2m(k)}, x_{2n(k)}, t),$$

$$M(x_{2m(k)-1}, x_{2n(k)}, t) \geq M(x_{2m(k)}, x_{2n(k)}, t) * M(x_{2m(k)}, x_{2m(k)-1}, t).$$

Using (13) and (14), we have

$$\lim_{n\to\infty} M(x_{2m(k)-1}, x_{2n(k)}, t) = 1 * \grave{o} = \lim_{n\to\infty} M(x_{2m(k)}, x_{2n(k)+1}, t). \tag{16}$$

Also,

$$M(x_{2m(k)-1}, x_{2n(k)+1}, t) \geq M(x_{2n(k)}, x_{2n(k)+1}, t) * M(x_{2m(k)-1}, x_{2m(k)}, t). \tag{17}$$

Using (13) and (16), we have

$$\lim_{n\to\infty} M(x_{2m(k)-1}, x_{2n(k)+1}, t) = \grave{o}. \tag{18}$$

Again, using (14)-(18), we have $\lim_{n\to\infty} M(x_{2m(k)-1}, x_{2n(k)}, t) = \grave{o}$.

Putting $x = x_{2m(k)-1}$, $y = x_{2n(k)}$ in (11), one can get

$$\psi(M(x_{2m(k)}, x_{2n(k)+1}, t)) = \psi(M(Tx_{2m(k)-1}, Sx_{2n(k)}, t))$$
$$\geq \psi(M(x_{2m(k)-1}, x_{2n(k)}, t)) + \phi(M(x_{2m(k)-1}, x_{2n(k)}, t)),$$

letting $k \to \infty$ and using (16) and (18), we get

$$\psi(\grave{o}) \geq \psi(\grave{o}) + \phi(\grave{o}),$$

this is contradiction with $\grave{o} > 0$. Thus $\{x_{2n}\}$ is a Cauchy sequence and hence $\{x_n\}$ is a Cauchy sequence. In complete metric space X , there exists z such that $x_n \to z$ as $n \to \infty$.

Let us now prove that z is a fixed point for T and S .

$$\psi(M(Tz, x_{2n(k)+1}, t)) = \psi(M(Tz, Sx_{2n(k)}, t))$$
$$\geq \psi(N(z, x_{2n(k)}, t)) + \phi(N(z, x_{2n(k)}, t)),$$

using the same argument as in Theorem 2.1 (eq- 9)and letting $n \to \infty$, we obtain

$$\psi(M(Tz, z, t)) \geq \psi(M(Tz, z, t)) + \phi(M(Tz, z, t)),$$

this implies $\psi(M(Tz, z, t)) = 1$. Hence $M(Tz, z, t) = 1$ gives that z is a fixed point of T .

Thus, we have

$$\psi(M(z, Sz, t)) = \psi(M(Tz, Sz, t))$$
$$\geq \psi(N(z, z, t)) + \phi(N(z, z, t))$$
$$= \psi(M(z, Sz, t)) + \phi(M(z, Sz, t)),$$

this implies $\psi(M(Sz, z, t)) = 1$. Hence $M(Sz, z, t) = 1$ or $Sz = z$.

To prove uniqueness, we consider another fixed point $w \in X$, then

$$\psi(M(z, w, t)) = \psi(M(Tz, Sw, t))$$
$$\geq \psi(N(z, w, t)) + \phi(N(z, w, t))$$
$$= \psi(M(z, w, t)) + \phi(M(z, w, t))$$

thus $\psi(M(z, w, t)) = 1.$

Hence $M(z, w, t) = 1$ or $z = w.$ This completes the proof.

Corollary 2.2 Let $(X, M, *)$ be a complete fuzzy metric space and $T : X \to X$ be a mapping such that for all $x, y \in X$,

$$\psi(M(Tx,Ty,t)) \geq \psi(N(x,y,t)) + \phi(N(x,y,t)), \quad (19)$$

where

1. $\psi : [0,1] \to [0,1]$ is a continuous monotone non-decreasing function with $\psi(t) = 1$ iff $t = 1$,
2. $\phi : [0,1] \to [0,1]$ is a upper semi-continuous function $\phi(t) > 0$ for $t \in (0,1)$ and $\phi(1) = 0$, and

$N(x, y, t) = min\{ M(x, y, t), M(Tx, x, t), M(Ty, y, t), M(y, Tx, t) * M(x, Ty, t)\}$,

then there exists a unique $u \in X$ such that $u = Tu$.

Example 2.2 Let $(X, M, *)$ be a complete fuzzy metric space with metric $d(x, y) = |x - y|$ and $X = [0,1]$. Let

$Tx = \frac{x}{2}$ *and* $Sx = 0$ for each $x \in [0,1]$ Then

$$N(x,y,t) = \min\left\{ \frac{t}{t+|x-y|}, \frac{t}{t+\frac{x}{2}}, \frac{t}{t+y}, \frac{t}{t+|x-y|} * \frac{t}{t+|x|} \right\},$$

$$= \begin{cases} \dfrac{t}{t+|x-y|} & \dfrac{x}{2} \leq y \leq x \\ \dfrac{t}{t+y} & y \geq x. \end{cases}$$

For $\phi(t) = t$ and $\psi(t) = 3t$, it is easy to show that

$\psi(M(Tx,Sy,t)) \geq \psi(N(x,y,t)) + \phi(N(x,y,t)),$ for all $x, y \in X$.

All the conditions of Theorem 2.2 are satisfied.

3. Conclusions

In this paper, the use of control functions has renewed the possibility of establishing new results in fuzzy metric fixed point theory. We extend the existing result in metric space towards fuzzy metric space with a new approach of using control functions.

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