



## Optimizing multi-skill call center staffing using queuing models: A study of service level

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**Abstract:** This paper addresses the staffing problem of a multi-skill call center with prioritized and patient customers based on a queuing model for minimizing staffing costs subject to service level requirements and determining the optimal number of agents. The model includes three types of calls and four groups of agents with different skills. Initially, a method for describing the state space is presented, followed by obtaining state-transition rates of the study model from  $M/M/c$  and  $M/M/c/c$  queuing systems. Equations for steady-state probabilities and service level computation are derived. A staffing calculation model for optimal agent numbers in each group is then formulated. An algorithm is proposed to find solutions, and its process is outlined. Finally, a numerical example is provided to analyze the system's sensitivity to several factors.

**Keywords:** Queuing model, multi-skill call center, service level, staffing problem

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## 1. Introduction

Call centers are the preferred and important way for many companies to communicate with their customers. The call center industry has undergone rapid growth in recent years. Call centers are widely utilized in many fields, including e-business, finance, banking, hotels, credit card companies, information centers, care hotlines, airlines, emergency services, telemarketing, help desks, and various other domains. A call center is characterized as a service system in which agents (servers) serve customers (callers) through telephone, fax, email, network communication software, as well as other multimedia channels. This fast progress in technology, with the variety of contact methods and the size of call centers, made the agents busier, their work more complicated, and required them to master many skills. However, it is important to note that each individual agent is unable to acquire proficiency in all skills. This shift has led to the transformation of numerous call centers from a single-skill model to a multi-skill approach. In multi-skill call centers, incoming calls may be categorized into distinct types, and agents are differentiated based on the specific types of calls they are capable of managing. Gans et al. (2003) illustrated an exemplary example of an international call center adept at managing calls in diverse languages. The study addressed multi-skill call center scenarios, specifically those involving a limited number of skill groups—three groups or fewer. Furthermore, the research offered thorough insights into various design approaches, encompassing V-design, M-design, W-design, and N-design. Ormeci (2004) conducted research on the dynamic admission control problem within an M-design multi-skill call center, examining scenarios involving two distinct call classes and three stations. Bhulai et al. (2008), Cezik and L'Ecuyer (2008) studied the cases of more call types and more skill groups of the multi-skill call centers.

The increasing significance and intricacy of contemporary call centers have led to a surge in literature dedicated to them, often concentrating on queuing models. In the context of a call center queuing model, the call agents are analogous to servers, while the calls represent customers. This theoretical framework is instrumental in tackling various challenges, including the efficient management of customer wait times, optimization of agent utilization, and meeting service level agreements. For more related surveys, we refer to the following works (Gans & Zhou, 2003; Koole & Mandelbaum, 2002; Kuhl et al., 2005). Aitchanov et al. (2021) presented solutions for optimizing contact centers, addressing infrastructure and financial challenges, with a focus on improving customer satisfaction by examining factors related to employees. Their study offers practical insights into call center management models and the scalability of infrastructure for emergency situations.

Call centers play a crucial role in the service industry, where the adept handling of customer inquiries and requests is essential for ensuring customer satisfaction and overall business prosperity. Efficiently operating call centers must strike a balance between the cost of service and service levels. It is imperative for call centers to ensure the presence of an appropriate number of agents possessing the requisite skills at any given time, aligning with the anticipated demand level. This challenge is commonly known as the staffing problem. A comprehensive overview of staffing problems was given (Aksin et al., 2007; Defraeye & Van Nieuwenhuysse, 2016; Koole & Pot, 2006; Wallace & Witt, 2005). In (Li et al., 2020; Priyai & Rajendran, 2018), the staffing problem is discussed in an M-design multi-skill call center with impatient customers. Also, Li and Yue (2016) conducted a study on multi-skill call centers with N-design, which involves partitioning agents into distinct groups based on their skills and utilizing skill-based routing to direct customers to appropriate agents. Dam et al. (2022) addressed staffing optimization in multi-skill call centers, aiming to minimize costs while meeting quality of service (QoS) targets. They introduced a joint chance-constrained approach that considers correlations between call types and employed a combination of heuristics and simulation-based methods to find staffing solutions, showing its effectiveness in complex call center scenarios with multiple call types and agent groups. Chevalier (2004) presented the staffing problem of a block model. Other research about this problem, see, e.g., (Liao et al., 2012).

Several researchers paid attention to applying heuristic methods to solve the staffing problem. Atlason et al. (2004) introduced an iterative cutting plane method for minimizing staffing costs in a single-skill call center model. Horng and Lin (2020) proposed a method that combines ordinal optimization with elephant herding optimization to efficiently address staffing problems, providing a viable solution within a reasonable computation time. In Lu et al. (2023), a personnel scheduling problem is described, and uncertain factors are analyzed and modelled. Avramidis et al. (2009) improved two-stage heuristic staffing search methods in the first stage and local search techniques in the second stage for solution refinement. Through a comparison with the cutting-plane method, the researchers determined that their heuristic not only yielded superior solutions but also resulted in faster execution. Pot et al. (2008) introduced a straightforward heuristic approach with local search for staffing in a call center. A comparative analysis with the cutting plane method led to the conclusion that the heuristic exhibited several desirable characteristics, including short computation time, high accuracy, and ease of implementation. Li et al. (2017) introduced a search algorithm grounded in the optimal computing budget allocation method. Through application in

a financial call center case study, the suggested approach demonstrated enhanced search efficiency and yielded satisfactory outcomes.

The rest of this paper is organized as follows: Section 2 describes a system model. Section 3 provides the state space, the state-transition rates, the steady-state probabilities, and the service level formula. Section 4 outlines the formulation of the staffing problem and discusses the proposal of the adapted algorithm to solve the problem, and Section 5 presents a numerical example. Finally, Section 6 concludes the paper.

## 2. Model description

In our model, the calls are categorized into three types (Type 1, Type 2, and Type 3) and agents are split into four groups (Group 1, Group 2, Group 3, and Group 4) with various skills. This queuing model is fully characterized by customer profiles (arrival process), agent properties (service process), routing policies, and the limitation of the waiting queue (queuing discipline). The model is shown in Figure 1.

### 2.1. Customer profiles

Three call types arrive according to the Poisson process with rates  $\lambda_1$ ,  $\lambda_2$ , and  $\lambda_3$  respectively. There are three queues (Queue 1, Queue 2, and Queue 3), the arriving calls are lined with it, which contain calls of Type 1, Type 2, and Type 3, respectively. There are infinite waiting spaces for three queues.

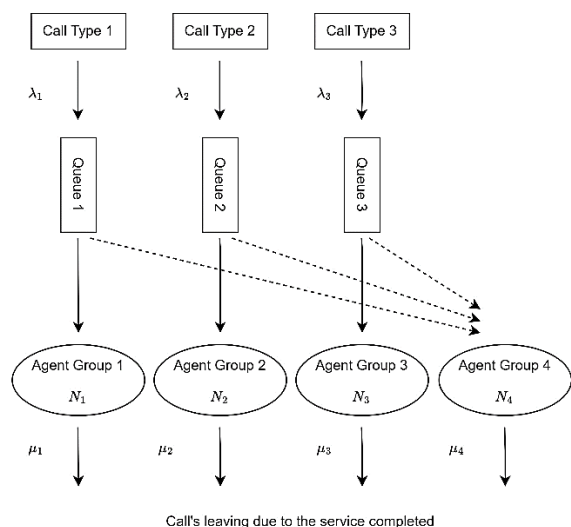


Figure 1. The multi-skill call center queue model.

### 2.2. Agent properties

There are four categories of agents. Group 1 (has Skill 1), Group 2 (has Skill 2), Group 3 (has Skill 3), and Group 4 (have Skill 1, Skill 2, and Skill 3) consist of  $N_1$ ,  $N_2$ ,  $N_3$ , and  $N_4$  agents,

respectively. Specialized agents  $N_1$ ,  $N_2$  and  $N_3$  can only serve calls of Type 1, Type 2, and Type 3 respectively while a flexible agent  $N_4$  can serve calls of both Type 1, Type 2, and Type 3. The service times of agents in Group 1, Group 2, Group 3, and Group 4 are all exponentially distributed with means  $\mu_1$ ,  $\mu_2$ ,  $\mu_3$ , and  $\mu_4$ , respectively.

### 2.3. Limitation of the waiting queue

The waiting spaces for three queues are infinite. Each type of call has its own queue. The queues of both three call types are independent of each other. For the same type of waiting calls, they are served by a free agent of its own group (Group 1, Group 2, or Group 3) and, a free agent in Group 4 according to FCFS (First-come First-served) discipline.

### 2.4. Routing policy

The routing policy in our model is skill-based routing and the importance of the three distinct types of calls. It is assumed that calls of Type 1 are more important than calls of Type 2 and Type 3. Also, the calls of Type 2 are more important than calls of Type 3. In other words, the calls of Type 1 have non-preemptive calls of Type 2 and Type 3; and the calls of Type 2 have non-preemptive calls of Type 3. The waiting calls (customers) of Type 1 are served by a free agent in Group 4; the waiting calls of Type 2 are served by a free agent in Group 4 if there are no waiting calls of Type 1 also, the waiting calls of Type 3 are served by a free agent in Group 4 if there are not waiting calls of Type 1 and Type 2.

## 3. Steady-state probabilities

In this section, we first determined the states of our model, then procured the state-transition rates in two cases (call arrival and service completion) and found the equilibrium equations for the steady-state probabilities of the model when the model is stationary. Finally, we computed the service level.

### 3.1. State space description

We have three groups with various skills in our model wherein each of them there are three states (idle, busy, and overload), while the fourth group has two states (idle, and busy).

I- An idle state: In this case, at least one agent is idle. This state is symbolized by 1.

II- A busy state: In this case, all agents in the group are busy and there are no calls waiting for service served by this group. This state is symbolized by 2.

III- An overload state: In this case, all agents in the group are busy, and there is at least one call waiting for service by this group. This state is symbolized by 3.

We note that the state space of the system model consisted of 34 states according to the routing policy assumed above. Thus, the state space is given by:

$$E = \{(1\ 1\ 1\ 1), (1\ 1\ 2\ 1), (1\ 1\ 2\ 2), (1\ 1\ 3\ 2), (1\ 2\ 1\ 1), (1\ 2\ 1\ 2), (1\ 2\ 2\ 2), (1\ 2\ 2\ 1), (1\ 2\ 3\ 2), (1\ 3\ 1\ 2), (1\ 3\ 2\ 2), (1\ 3\ 3\ 2), (2\ 1\ 1\ 1), (2\ 1\ 1\ 2), (2\ 1\ 2\ 1), (2\ 1\ 2\ 2), (2\ 1\ 3\ 2), (2\ 2\ 1\ 1), (2\ 2\ 1\ 2), (2\ 2\ 2\ 2), (2\ 2\ 2\ 1), (2\ 2\ 3\ 2), (2\ 3\ 1\ 2), (2\ 3\ 2\ 2), (2\ 3\ 3\ 2), (3\ 1\ 1\ 2), (3\ 1\ 2\ 2), (3\ 1\ 3\ 2), (3\ 2\ 1\ 2), (3\ 2\ 2\ 2), (3\ 2\ 3\ 2), (3\ 3\ 1\ 2), (3\ 3\ 2\ 2), (3\ 3\ 3\ 2)\}$$

let  $S_i, (i = 1, 2, 3, \dots, 34)$  be the  $i$ th state in the state space  $E$ . Let  $n_m, (m = 1, 2, 3)$  be the number of calls waiting in a queue that are assumed to be served by Group  $m$  plus the busy agents in Group  $m$ , while  $n_4$  be the number of calls being serviced by agents of Group 4 as no waiting calls in the queue according to routing policy.

### 3.2. The construction of the state-transition rates

We derive the state-transition rates by using the results of  $M/M/c$  and  $M/M/c/c$  queueing systems (which are given by Shortle et al. (2018)). There are only two events that can make the state transfer: call arrival or service fulfillment. We will be debating the two cases separately to obtain how the state transition rates are computed.

#### 3.2.1. The transfer of states due to the call arrival

We assume the state  $S_1 = (1\ 1\ 1\ 1)$ , and  $S_{13} = (2\ 1\ 1\ 1)$ , for example. The diagram of the transition from state  $S_1$  to state  $S_{13}$  is shown in Figure 2.

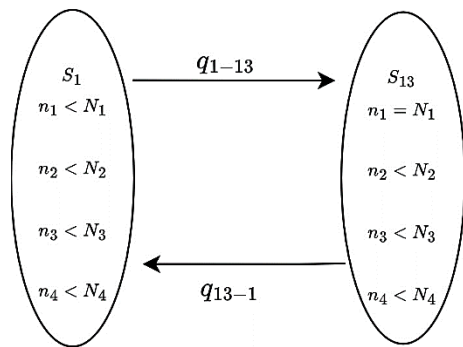


Figure 2. Diagram of the state-transition rate from the state  $S_1$  to the state  $S_{13}$ .

Let  $q_{(i-j)}, (i, j = 1, 2, 3, \dots, 34)$  denote the state-transition rate. The trigger for the transfer from the state  $S_1$  to the state  $S_{13}$  is obtained as follows:

$$q_{1-13} = \lambda_1 P(n_1 = N_1 - 1), \tag{1}$$

where  $P(n_1 = N_1 - 1)$  is the probability that the number of call Type 1 needing to be serviced by the agents in

Group 1 is  $n_1 = N_1 - 1$  for state  $S_1$ . Note that  $n_1 < N_1, n_2 < N_2,$  and  $n_3 < N_3$  (i.e., if the operation is in state  $S_1$  then the number of calls of either Type 1 or Type 2 or Type 3 is less than the number of agents either in Group 1 or Group 2 or Group 3), and that the three queues are independent of each other so the results of the  $M/M/c/c$  loss queuing system can be used. Consequently, we have

$$P(n_1 = N_1 - 1) = \frac{\rho_1^{N_1-1}}{(N_1 - 1)! \sum_{j=0}^{N_1} \frac{\rho_1^j}{j!}}, \tag{2}$$

Where

$$\rho_1 = \frac{\lambda_1}{\mu_1}, a_1 = \frac{\lambda_1}{N_1 \mu_1}.$$

In a comparable way, we can acquire the other transition rates  $q_{(i-j)}$  occurred by call arrival as shown in Addendum A.1 of Appendix A.

#### 3.2.2. The transfer of states due to the fulfillment of service

We assume the state  $S_2 = (1\ 1\ 2\ 1)$ , and  $S_1 = (1\ 1\ 1\ 1)$ , for example. If the call of Type 3 finished the service, then the set of states will be varied from state  $S_2$  to  $S_1$ . The diagram of the transition from state  $S_2$  to state  $S_1$  is shown in Figure 3.

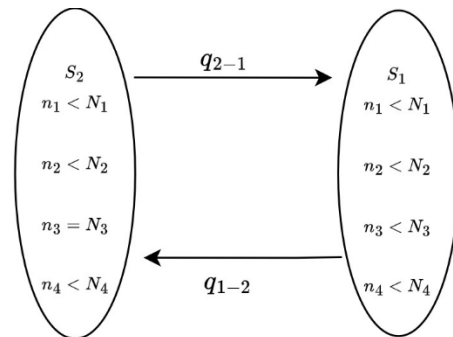


Figure 3. Diagram of the state-transition rate from the state  $S_2$  to the state  $S_1$ .

In the state  $S_2, n_3 = N_3$  and  $N_3 \mu_3$  is the service rate for the call of Type 3. The trigger for the transfer from state  $S_2$  to  $S_1$  is obtained as follows:

$$q_{2-1} = N_3 \mu_3. \tag{3}$$

In a comparable way, we can acquire the other transition rates  $q_{(i-j)}$  caused by the completion of service of a call as shown in Addendum A.2 of Appendix A.

The results of the  $M/M/c$  queueing system can be used to obtain the probabilities of  $P(n_1 = N_1 + 1), P(n_2 = N_2 + 1),$  and  $P(n_3 = N_3 + 1)$  as follows:

$$P(n_1 = N_1 + 1) = \frac{\rho_1^{N_1+1}}{N_1 (N_1)!} P_0^1, \tag{4}$$

$$P(n_2 = N_2 + 1) = \frac{\rho_2^{N_2+1}}{N_2 (N_2)!} P_0^2, \tag{5}$$

$$P(n_3 = N_3 + 1) = \frac{\rho_3^{N_3+1}}{N_3 (N_3)!} P_0^3, \tag{6}$$

where

$$P_0^1 = \left[ \sum_{j=0}^{N_1-1} \frac{\rho_1^j}{j!} + \frac{N_1 \rho_1^{N_1}}{N_1! (N_1 - \rho_1)} \right]^{-1},$$

$$P_0^2 = \left[ \sum_{j=0}^{N_2-1} \frac{\rho_2^j}{j!} + \frac{N_2 \rho_2^{N_2}}{N_2! (N_2 - \rho_2)} \right]^{-1},$$

$$P_0^3 = \left[ \sum_{j=0}^{N_3-1} \frac{\rho_3^j}{j!} + \frac{N_3 \rho_3^{N_3}}{N_3! (N_3 - \rho_3)} \right]^{-1}.$$

### 3.3. The computation of steady-state probabilities

The steady-state probabilities of the state procedure are denoted by  $P_i, i = 1, 2, 3, \dots, 34$  and are shown for each state by applying the equilibrium equation of the system (rate in = rate out) as follows:

$$P_1 X_1 = X_2, P_2 X_3 = X_4, P_3 X_5 = X_6, \tag{7}$$

$$P_4 X_7 = X_8, P_5 X_9 = X_{10}, P_6 X_{11} = X_{12}, \tag{8}$$

$$P_7 X_{13} = X_{14}, P_8 X_{15} = X_{16}, P_9 X_{17} = X_{18}, \tag{9}$$

$$P_{10} X_{19} = X_{20}, P_{11} X_{21} = X_{22}, P_{12} X_{23} = X_{24}, \tag{10}$$

$$P_{13} X_{25} = X_{26}, P_{14} X_{27} = X_{28}, P_{15} X_{29} = X_{30}, \tag{11}$$

$$P_{16} X_{31} = X_{32}, P_{17} X_{33} = X_{34}, P_{18} X_{35} = X_{36}, \tag{12}$$

$$P_{19} X_{37} = X_{38}, P_{20} X_{39} = X_{40}, P_{21} X_{41} = X_{42}, \tag{13}$$

$$P_{22} X_{43} = X_{44}, P_{23} X_{45} = X_{46}, P_{24} X_{47} = X_{48}, \tag{14}$$

$$P_{25} X_{49} = X_{50}, P_{26} X_{51} = X_{52}, P_{27} X_{53} = X_{54}, \tag{15}$$

$$P_{28} X_{55} = X_{56}, P_{29} X_{57} = X_{58}, P_{30} X_{59} = X_{60}, \tag{16}$$

$$P_{31} X_{61} = X_{62}, P_{32} X_{63} = X_{64}, P_{33} X_{65} = X_{66}, \tag{17}$$

$$P_{34} X_{67} = X_{68}, \tag{18}$$

$$\sum_{i=1}^{34} P_i = 1. \tag{19}$$

Remark. The description of the  $X_y, y = 1, 2, 3, \dots, 68$  is mentioned in Addendum A.3 of Appendix A.

We note that “Equation 19” means the total probability of being in any of the steady-state states is equal to 1. Since the above equations are linear, so by solving these equations, (using MATLAB software), we can obtain all the steady-state probabilities.

### 3.4. The computation of service level

Calculating service level is used in our paper to evaluate the performance of a multi-skill call center and is calculated by using steady-state probabilities. The service level can be expressed as the percentage of the calls that should be serviced within a given waiting time  $T_i$ , denoted as  $\frac{P_i}{T_i}$ .

Let  $P_{st}^i = 1 - P_{ns}^i, i = 1, 2, 3$  be the probability that the call of Type  $i$  is serviced in a fixed time  $T_i$ .

Consider the call of Type 1, for example. We suppose that the service level of the call of Type 1 is known as the probability of the calls are not serviced in a fixed time  $T_1$ . Calls of Type 1 has a queue only occur in the states  $S_{26} = (3 1 1 2), S_{27} = (3 1 2 2), S_{28} = (3 1 3 2), S_{29} = (3 2 1 2), S_{30} = (3 2 2 2), S_{31} = (3 2 3 2), S_{32} = (3 3 1 2), S_{33} = (3 3 2 2),$  and  $S_{34} = (3 3 3 2)$ . The service rate for the call of Type 1 in each state of  $S_{26}, S_{27}, S_{28}, S_{29}, S_{30}, S_{31}, S_{32}, S_{33},$  and  $S_{34}$ . is  $N_1 \mu_1 + N_4 \mu_4$  as all agents in Group 1 are busy and there are waiting calls of Type 1 in a queue if an agent in Group 4 finished its service, an agent will select the call of Type 1 in the queue to serve instantly. Therefore, the number of calls that could be served in time  $T_1$  is  $T_1(N_1 \mu_1 + N_4 \mu_4)$ . We can obtain the probability of the call of Type 1 cannot be serviced in time  $T_1, P_{ns}^1$ , as follows:

$$P_{ns}^1 = W_1 \sum_{i=K_1}^{\infty} P(n_1 = i), \tag{20}$$

Where

$$W_1 = (P_{26} + P_{27} + P_{28} + P_{29} + P_{30} + P_{31} + P_{32} + P_{33} + P_{34}), \tag{21}$$

$$K_1 = N_1 + N_4 + T_1(N_1 \mu_1 + N_4 \mu_4). \tag{22}$$

Similarly, for a call of Type 2, calls have a queue only occur in the states  $S_{10} = (1 3 1 2), S_{11} = (1 3 2 2), S_{12} = (1 3 3 2), S_{23} = (2 3 1 2), S_{24} = (2 3 2 2)$  and  $S_{25} = (2 3 3 2)$ . Thus, we can obtain the probability of the call of Type 2 cannot be serviced in time  $T_2, P_{ns}^2$ , as follows:

$$P_{ns}^2 = W_2 \sum_{i=K_2}^{\infty} P(n_2 = i), \tag{23}$$

where

$$W_2 = (P_{10} + P_{11} + P_{12} + P_{23} + P_{24} + P_{25}), \quad (24)$$

$$K_2 = N_2 + N_4 + T_2(N_2\mu_2 + N_4\mu_4). \quad (25)$$

And, by the same analysis method above, we can obtain the probability of the call of Type 3 cannot be serviced in time  $T_3$ ,  $P_{ns}^3$ , as follows:

$$P_{ns}^3 = W_3 \sum_{i=K_3}^{\infty} P(n_3 = i), \quad (26)$$

where

$$W_3 = (P_4 + P_9 + P_{17} + P_{22}), \quad (27)$$

$$K_3 = N_3 + N_4 + T_3(N_3\mu_3 + N_4\mu_4). \quad (28)$$

And,  $P(n_1 = i)$ ,  $P(n_2 = i)$ , and  $P(n_3 = i)$  are the probabilities that there are  $i$  customers in the  $M/M/N_1$ ,  $M/M/N_2$ , and  $M/M/N_3$  queueing system with arrival rate  $\lambda_1$ ,  $\lambda_2$ , and  $\lambda_3$  and service rates  $\mu_1$ ,  $\mu_2$ , and  $\mu_3$ , respectively, which their formulas are given in (Shortle et al., 2018). By using MATLAB software, we can get the service levels  $P_{sl}^1$ ,  $P_{sl}^2$ , and  $P_{sl}^3$ .

#### 4. Staffing optimization

We offered the optimization model of the staffing problem to find the optimal number of agents in each group to minimize the cost of the system while ensuring that the service level for each call type meets the minimum service level requirement. The service rate of each call type is a key factor in determining the optimal number of agents in each group. Supposing that the cost of the agents' Group 1 is  $C_1$ , the cost of the agents' Group 2 is  $C_2$ , the cost of the agents' Group 3 is  $C_3$ , and the cost of the agents' Group 4 is  $C_4$ . To minimize the system models' cost, we seek to get the optimal number of agents  $N_1$ ,  $N_2$ ,  $N_3$ , and  $N_4$ . The staffing optimization can be expressed as follows:

$$\begin{aligned} \min z &= C_1N_1 + C_2N_2 + C_3N_3 + C_4N_4, \\ \text{s. t.} \quad & P_{sl}^1 \geq \alpha_1, \\ & P_{sl}^2 \geq \alpha_2, \\ & P_{sl}^3 \geq \alpha_3, \\ & a \leq N_i \leq b, a, b, N_i \in Z^+; i = 1, 2, 3, 4. \end{aligned} \quad (29)$$

where  $\alpha_1$ ,  $\alpha_2$ , and  $\alpha_3$  are the given service rate of the call Type 1, the call Type 2, and the call Type 3, respectively,  $Z^+$  denote the set of positive integers. The number of agents within each group may be chosen within the interval of  $(a, b)$ . This problem is nonlinear integer programming with a linear objective function. The optimization involves determining four

variables, namely,  $N_1$ ,  $N_2$ ,  $N_3$ , and  $N_4$ . The constraints pertain to incoming calls aimed at meeting specific service level criteria. Notably, the constraints are intricately nonlinear, as evident from the service level formula. Given the involvement of four variables in this model, an effective resolution can be achieved through the application of the following algorithm using MATLAB software to obtain the optimal numbers of every agent group and the minimum cost associated with it.

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#### Algorithm 1 Staff Problem Algorithm

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**Inputs:**  $C_1, C_2, C_3, C_4, a, b, \alpha_1, \alpha_2, \alpha_3, \lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3, \mu_4, T_1, T_2, T_3$ .

**Outputs:** The optimal number of agents in each group and minimum cost.

**Step 1:** Initialize the parameters.

**Step 2:** Select all variable combinations of  $N_i$ 's such that  $a \leq N_i \leq b; i = 1, 2, 3, 4$ .

**Step 3:** Compute all transition rates (from Eqn. (A.1) to (A.89)), the steady-state probabilities (from Eqn. (7) to (18)), and then the service levels (from Eqn. (20), (23), and (26)).

**Step 4:** Compare  $P_{sl}^i$  and  $\alpha_i; i = 1, 2, 3$ .

(a) If  $P_{sl}^i \geq \alpha_i$ , go to step (5).

(b) If  $P_{sl}^i < \alpha_i$  for any  $i$ , go to step (2).

**Step 5:** Determine the number of agents in each group and compute the cost from the Eqn.

$\min Z = C_1N_1 + C_2N_2 + C_3N_3 + C_4N_4$ .

**Step 6:** Repeat step (2) until all selections of variable combinations of  $N_i$ 's occur.

**Step 7:** Compare all previous cost in step (5) and then find the minimum cost,  $N_1, N_2, N_3, \text{ and } N_4$ .

**EXIT.**

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#### 5. Example

The purpose of this example is to indicate how these factors, like service levels and the number of agents in each group, affect the entire system. We consider the same system model introduced earlier in the above sections, but with the following parameters. Arrival rates are  $\lambda_1 = 5$ ,  $\lambda_2 = 4$ , and  $\lambda_3 = 3$ , and service rates are  $\mu_1 = 0.8$ ,  $\mu_2 = 0.6$ ,  $\mu_3 = 0.4$ ,  $\mu_4 = 0.2$ ,  $\alpha_1 = 0.8$ ,  $\alpha_2 = 0.7$ , and  $\alpha_3 = 0.6$ . The fixed waiting times are  $T_1 = 20$ ,  $T_2 = 30$ , and  $T_3 = 40$ . The cost for each agent in the four groups are  $C_1 = 25$ ,  $C_2 = 20$ ,  $C_3 = 15$ , and  $C_4 = 10$ . To show how the results in Tables (1, 2, 3) were obtained, let us take case (1), for example, when  $N_1 = 21$ ,  $N_2 = 16$ ,  $N_3 = 8$ , and  $N_4 = 5$ . By giving the MATLAB software, the parameter settings are as follows:  $\lambda_1 = 5$ ,  $\lambda_2 = 4$ ,  $\lambda_3 = 3$ ,  $\mu_1 = 0.8$ ,  $\mu_2 = 0.6$ ,  $\mu_3 = 0.4$ ,  $\mu_4 = 0.2$ ,  $T_1 = 20$ ,  $T_2 = 30$ ,  $T_3 = 40$ ,  $C_1 = 25$ ,  $C_2 = 20$ ,  $C_3 = 15$ ,  $C_4 = 10$ ,  $(a, b) = (2, 100)$  and utilizing "Equations from (A.1) to (A.89)" we get all the state-transition rates of both call arrival and service fulfilment, then by substituting in "Equations from (7) to (18)", we get all the steady-state probabilities, and by using "Equations from (20) to (28)", we calculate the service levels, and from "Equation (29)" and applying the above algorithm, we get the optimal number of agents in each group. By the same manner, we can get the other cases.

Based on the results in Table (1), we can see that all steady-state probabilities are present in small-sized cases (50 or less agents), while in medium (51-200 agents) or large (more than 201 agents)-sized call centers, some steady-state probabilities are absent in each case, indicating a reduction in the complexity

of the system as it scales. Comparing probabilities across cases reveals patterns and variations in system behavior, providing insights into resource utilization and performance.

Table 1. Numerical results of the steady-state probabilities.

Case	1	2	3	4	5	6
$P_1$	0.0474	0.0488	0.6318	0.5923	0.7455	0.6453
$P_2$	0.0183	0.0203	0.0000	0.0000	0.0000	0.0000
$P_3$	0.0136	0.0137	0.0000	0.0000	0.0000	0.0000
$P_4$	0.5229	0.5918	0.2651	0.2983	0.1982	0.3066
$P_5$	0.0017	0.0012	0.0000	0.0000	0.0000	0.0000
$P_6$	0.0025	0.0018	0.0000	0.0000	0.0000	0.0000
$P_7$	0.0013	0.0009	0.0000	0.0000	0.0000	0.0000
$P_8$	0.0016	0.0014	0.0000	0.0000	0.0000	0.0000
$P_9$	0.0030	0.0022	0.0000	0.0000	0.0000	0.0000
$P_{10}$	0.0355	0.0269	0.0186	0.0211	0.0081	0.0079
$P_{11}$	0.0067	0.0051	0.0000	0.0000	0.0000	0.0000
$P_{12}$	0.1626	0.1272	0.0423	0.0453	0.0283	0.0268
$P_{13}$	0.0006	0.0005	0.0000	0.0000	0.0000	0.0000
$P_{14}$	0.0012	0.0010	0.0000	0.0000	0.0000	0.0000
$P_{15}$	0.0008	0.0009	0.0000	0.0000	0.0000	0.0000
$P_{16}$	0.0006	0.0005	0.0000	0.0000	0.0000	0.0000
$P_{17}$	0.0013	0.0012	0.0000	0.0000	0.0000	0.0000
$P_{18}$	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
$P_{19}$	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000
$P_{20}$	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000
$P_{21}$	0.0003	0.0003	0.0000	0.0000	0.0000	0.0000
$P_{22}$	0.0004	0.0004	0.0000	0.0000	0.0000	0.0000
$P_{23}$	0.0004	0.0003	0.0000	0.0000	0.0000	0.0000
$P_{24}$	0.0005	0.0004	0.0000	0.0000	0.0000	0.0000
$P_{25}$	0.0006	0.0005	0.0000	0.0000	0.0000	0.0000
$P_{26}$	0.0221	0.0192	0.0100	0.0113	0.0031	0.0024
$P_{27}$	0.0040	0.0035	0.0000	0.0000	0.0000	0.0000
$P_{28}$	0.0923	0.0854	0.0199	0.0191	0.0106	0.0073
$P_{29}$	0.0005	0.0004	0.0000	0.0000	0.0000	0.0000
$P_{30}$	0.0005	0.0004	0.0000	0.0000	0.0000	0.0000
$P_{31}$	0.0007	0.0005	0.0000	0.0000	0.0000	0.0000
$P_{32}$	0.0157	0.0118	0.0042	0.0045	0.0014	0.0008
$P_{33}$	0.0017	0.0013	0.0000	0.0000	0.0000	0.0000
$P_{34}$	0.0381	0.0300	0.0082	0.0080	0.0048	0.0030

Table (2) demonstrates that the service level formula is applicable to call centers of any scale. In multi-skill call centers, the total number of agents remains constant, but the composition of agent groups significantly impacts the overall service level. Hence, optimizing the numbers of each agent group is crucial to achieve the desired service level while minimizing system costs.

Table 2. Numerical results of the service levels.

Case	N	$N_1$	$N_2$	$N_3$	$N_4$	$P_{sl}^1$	$P_{sl}^2$	$P_{sl}^3$
1	48	21	15	8	4	0.8242	0.7940	0.7265
2	48	21	17	8	2	0.8474	0.8396	0.6565
3	140	35	35	35	35	0.9577	0.9392	0.7349
4	140	40	35	35	30	0.9570	0.9336	0.7017
5	220	40	40	40	100	0.9801	0.9635	0.8018
6	220	70	50	50	50	0.9865	0.9653	0.6934

As shown in Table (3). The table shows that the optimal number of agents in each group varies for each case, and it also affects the system cost. It demonstrates that the system cost can be minimized by adjusting the number of agents in each group based on the service rate and cost for each group.

Table 3. Numerical results of the optimal  $N_1, N_2, N_3,$  and  $N_4$ .

Case	$N_1$	$N_2$	$N_3$	$N_4$	Cost
1	21	15	8	4	985
2	21	17	8	2	1005
3	35	35	35	35	2450
4	40	35	35	30	2525
5	40	40	40	100	3400
6	70	50	50	50	4000

### 6. Conclusion

The paper focuses on a multi-skill call center that operates on a queuing model, specifically the exponential model. The call center has three types of customers and four groups of agents, with specialized agents attending to their respective customer type and flexible agents managing both types. By examining the state space of the system and using results from M/M/c/c and M/M/c queuing systems, the transition rates of the state sets were determined. Equilibrium equations for the system's steady-state equations were also formulated. The paper presents a formula for computing the service level and discusses the technique used to calculate the optimal number of agents in each group to solve the staffing problem. Overall, these results can provide insights and recommendations for call center managers to improve the service levels and optimize their staffing levels. The paper's methodology easily adapts to various multi-skill call centers and proves valuable for evaluating contact center performance. Future extensions could include analyzing non-exponential models with general service times, impatient times, and skill-based routing without priority.

### Conflict of interest

The authors have no conflict of interest to declare.

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## Appendix

### A.1. Addendum

In this addendum to Appendix A, we present the other transition rates  $q_{(i-j)}$  occurred by call arrival. The results are re-listed below.

$$q_{1-13} = q_{2-15} = q_{3-16} = q_{4-17} = q_{5-18} = q_{6-19} \\ = q_{7-20} = q_{8-21} = q_{9-22} = q_{10-23} = q_{11-24} \\ = q_{12-25} = \lambda_1 P(n_1 = N_1 - 1) \quad (A.1)$$

$$q_{1-5} = q_{2-8} = q_{3-7} = q_{4-9} = q_{13-18} = q_{14-19} \\ = q_{15-21} = q_{16-20} = q_{17-22} = q_{26-29} = q_{27-30} \\ = q_{28-31} = \lambda_2 P(n_2 = N_2 - 1) \quad (A.2)$$

$$q_{1-2} = q_{5-8} = q_{6-7} = q_{10-11} = q_{13-15} = q_{14-16} \\ = q_{18-21} = q_{19-20} = q_{23-24} = q_{26-27} = q_{29-30} \\ = q_{32-33} = \lambda_3 P(n_3 = N_3 - 1) \quad (A.3)$$

$$q_{14-26} = q_{16-27} = q_{17-28} = q_{19-29} = q_{20-30} \\ = q_{22-31} = q_{23-32} = q_{24-33} = q_{25-34} \\ = \lambda_1 \quad (A.4)$$

$$q_{6-10} = q_{7-11} = q_{9-12} = q_{19-23} = q_{20-24} = q_{22-25} \\ = q_{29-32} = q_{30-33} = q_{31-34} = \lambda_2 \quad (A.5)$$

$$q_{3-4} = q_{7-9} = q_{11-12} = q_{16-17} = q_{20-22} = q_{24-25} \\ = q_{27-28} = q_{30-31} = q_{33-34} = \lambda_3 \quad (A.6)$$

$$q_{1-18} = q_{2-21} = q_{3-20} = q_{4-22} \\ = \lambda_1 P(n_1 = N_1 - 1) + \lambda_2 P(n_2 = N_2 - 1) \quad (A.7)$$

$$q_{1-15} = q_{5-21} = q_{6-20} = q_{10-24} \\ = \lambda_1 P(n_1 = N_1 - 1) + \lambda_3 P(n_3 = N_3 - 1) \quad (A.8)$$

$$q_{1-8} = q_{13-21} = q_{14-20} = q_{26-30} \\ = \lambda_2 P(n_2 = N_2 - 1) + \lambda_3 P(n_3 = N_3 - 1) \quad (A.9)$$

$$q_{1-21} = \lambda_1 P(n_1 = N_1 - 1) + \lambda_2 P(n_2 = N_2 - 1) \\ + \lambda_3 P(n_3 = N_3 - 1) \quad (A.10)$$

$$q_{19-32} = q_{20-33} = q_{22-34} = \lambda_1 + \lambda_2 \quad (A.11)$$

$$q_{16-28} = q_{20-31} = q_{24-34} = \lambda_1 + \lambda_3 \quad (A.12)$$

$$q_{7-12} = q_{20-25} = q_{30-34} = \lambda_2 + \lambda_3 \quad (A.13)$$

$$q_{20-34} = \lambda_1 + \lambda_2 + \lambda_3 \quad (A.14)$$

$$q_{13-14} = \lambda_1 P^3(n_4 = N_4 - 1) \quad (A.15)$$

$$q_{5-6} = \lambda_2 P^1(n_4 = N_4 - 1) \quad (A.16)$$

$$q_{2-3} = \lambda_3 P^2(n_4 = N_4 - 1) \quad (A.17)$$

$$q_{18-23} = \lambda_1 P^3(n_4 = N_4 - 1) + \lambda_2 \quad (A.18)$$

$$q_{8-9} = \lambda_2 P^1(n_4 = N_4 - 1) + \lambda_3 \quad (A.19)$$

$$q_{15-17} = \lambda_1 P^3(n_4 = N_4 - 1) + \lambda_3 \quad (A.20)$$

$$q_{21-25} = \lambda_1 P^3(n_4 = N_4 - 1) + \lambda_2 + \lambda_3 \quad (A.21)$$

$$q_{6-23} = q_{7-24} = q_{9-25} = \lambda_1 P(n_1 = N_1 - 1) \\ + \lambda_2 \quad (A.22)$$

$$q_{3-17} = q_{7-22} = q_{11-25} = \lambda_1 P(n_1 = N_1 - 1) \\ + \lambda_3 \quad (A.23)$$

$$q_{7-25} = \lambda_1 P(n_1 = N_1 - 1) + \lambda_2 + \lambda_3 \quad (A.24)$$

$$q_{14-29} = q_{16-30} = q_{17-31} = \lambda_1 + \lambda_2 P(n_2 = N_2 - 1) \\ (A.25)$$

$$q_{3-9} = q_{16-22} = q_{27-31} = \lambda_2 P(n_2 = N_2 - 1) + \lambda_3 \quad (A.26)$$

$$q_{16-31} = \lambda_1 + \lambda_2 P(n_2 = N_2 - 1) + \lambda_3 \quad (A.27)$$

$$q_{14-27} = q_{19-30} = q_{23-33} = \lambda_1 + \lambda_3 P(n_3 = N_3 - 1) \quad (A.28)$$

$$q_{6-11} = q_{19-24} = q_{29-33} = \lambda_2 + \lambda_3 P(n_3 = N_3 - 1) \quad (A.29)$$

$$q_{19-33} = \lambda_1 + \lambda_2 + \lambda_3 P(n_3 = N_3 - 1) \quad (A.30)$$

$$q_{3-22} = \lambda_1 P(n_1 = N_1 - 1) + \lambda_2 P(n_2 = N_2 - 1) + \lambda_3 \quad (A.31)$$

$$q_{6-24} = \lambda_1 P(n_1 = N_1 - 1) + \lambda_2 + \lambda_3 P(n_3 = N_3 - 1) \quad (A.32)$$

$$q_{14-30} = \lambda_1 + \lambda_2 P(n_2 = N_2 - 1) \\ + \lambda_3 P(n_3 = N_3 - 1) \quad (A.33)$$

$$q_{5-19} = \lambda_1 P(n_1 = N_1 - 1) + \lambda_2 P^1(n_4 = N_4 - 1) \quad (A.34)$$

$$q_{2-16} = \lambda_1 P(n_1 = N_1 - 1) + \lambda_3 P^2(n_4 = N_4 - 1) \quad (A.35)$$

$$q_{2-7} = \lambda_2 P(n_2 = N_2 - 1) + \lambda_3 P^2(n_4 = N_4 - 1) \quad (A.36)$$

$$q_{13-19} = \lambda_1 P^3(n_4 = N_4 - 1) + \lambda_2 P(n_2 = N_2 - 1) \quad (A.37)$$

$$q_{5-7} = \lambda_2 P^1(n_4 = N_4 - 1) + \lambda_3 P(n_3 = N_3 - 1) \quad (A.38)$$

$$q_{13-16} = \lambda_1 P^3(n_4 = N_4 - 1) + \lambda_3 P(n_3 = N_3 - 1) \quad (A.39)$$

$$q_{2-20} = \lambda_1 P(n_1 = N_1 - 1) + \lambda_2 P(n_2 = N_2 - 1) \\ + \lambda_3 P^2(n_4 = N_4 - 1) \quad (A.40)$$

$$q_{5-20} = \lambda_1 P(n_1 = N_1 - 1) + \lambda_2 P^1(n_4 = N_4 - 1) \\ + \lambda_3 P(n_3 = N_3 - 1) \quad (A.41)$$

$$q_{13-20} = \lambda_1 P^3(n_4 = N_4 - 1) + \lambda_2 P(n_2 = N_2 - 1) \\ + \lambda_3 P(n_3 = N_3 - 1) \quad (A.42)$$

$$q_{8-22} = \lambda_1 P(n_1 = N_1 - 1) + \lambda_2 P^1(n_4 = N_4 - 1) \\ + \lambda_3 \quad (A.43)$$

$$q_{15-22} = \lambda_1 P^3(n_4 = N_4 - 1) + \lambda_2 P(n_2 = N_2 - 1) \\ + \lambda_3 \quad (A.44)$$

$$q_{18-24} = \lambda_1 P^3(n_4 = N_4 - 1) + \lambda_2 \\ + \lambda_3 P(n_3 = N_3 - 1) \quad (A.45)$$

$$\text{Where} \quad P(n_2 = N_2 - 1) = \frac{\rho_2^{N_2-1}}{(N_2 - 1)! \sum_{j=0}^{N_2} \frac{\rho_2^j}{j!}}, \quad (A.46)$$

$$\rho_2 = \frac{\lambda_2}{\mu_2}, a_2 = \frac{\lambda_2}{N_2 \mu_2}. \quad (A.46)$$

$$P(n_3 = N_3 - 1) = \frac{\rho_3^{N_3-1}}{(N_3 - 1)! \sum_{j=0}^{N_3} \frac{\rho_3^j}{j!}}, \quad (A.47)$$

$$\rho_3 = \frac{\lambda_3}{\mu_3}, a_3 = \frac{\lambda_3}{N_3 \mu_3}. \quad (A.47)$$

$$P^1(n_4 = N_4 - 1) = \frac{\rho_4^{N_4-1}}{(N_4 - 1)! \sum_{j=0}^{N_4} \frac{\rho_4^j}{j!}}, \quad (A.48)$$

$$\rho_4 = \frac{\lambda_2}{\mu_4}. \quad (A.48)$$

$$\rho_5 = \frac{\lambda_3}{\mu_4}$$

$$P^2(n_4 = N_4 - 1) = \frac{\rho_5^{N_4-1}}{(N_4 - 1)! \sum_{j=0}^{N_4} \frac{\rho_5^j}{j!}} \quad (A.49)$$

$$\rho_6 = \frac{\lambda_1}{\mu_4}$$

$$P^3(n_4 = N_4 - 1) = \frac{\rho_6^{N_4-1}}{(N_4 - 1)! \sum_{j=0}^{N_4} \frac{\rho_6^j}{j!}} \quad (A.50)$$

### A. 2. Addendum

In this addendum to Appendix A, we present the other transition rates  $q_{(i-j)}$  occurred by fulfillment of service of a call. The results are re-listed below.

$$q_{13-1} = q_{15-2} = q_{16-3} = q_{17-4} = q_{18-5} = q_{19-6} = q_{20-7} = q_{21-8} = q_{22-9} = q_{23-10} = q_{24-11} = q_{25-12} = N_1\mu_1 \quad (A.51)$$

$$q_{5-1} = q_{8-2} = q_{7-3} = q_{9-4} = q_{18-13} = q_{19-14} = q_{21-15} = q_{20-16} = q_{22-17} = q_{29-26} = q_{30-27} = q_{31-28} = N_2\mu_2 \quad (A.52)$$

$$q_{2-1} = q_{8-5} = q_{7-6} = q_{11-10} = q_{15-13} = q_{16-14} = q_{21-18} = q_{20-19} = q_{24-23} = q_{27-26} = q_{30-29} = q_{33-32} = N_3\mu_3 \quad (A.53)$$

$$q_{26-14} = q_{27-16} = q_{28-17} = q_{29-19} = q_{30-20} = q_{31-22} = q_{32-33} = q_{33-24} = q_{34-25} = N_1\mu_1 P(n_1 = N_1 + 1) \quad (A.54)$$

$$q_{10-6} = q_{11-7} = q_{12-9} = q_{23-19} = q_{24-20} = q_{25-22} = q_{32-29} = q_{33-30} = q_{34-31} = N_2\mu_2 P(n_2 = N_2 + 1) \quad (A.55)$$

$$q_{4-3} = q_{9-7} = q_{12-11} = q_{17-16} = q_{22-20} = q_{25-24} = q_{28-27} = q_{31-30} = q_{34-33} = N_3\mu_3 P(n_3 = N_3 + 1) \quad (A.56)$$

$$q_{18-1} = q_{21-2} = q_{20-3} = q_{22-4} = N_1\mu_1 + N_2\mu_2 \quad (A.57)$$

$$q_{15-1} = q_{21-5} = q_{20-6} = q_{24-10} = N_1\mu_1 + N_3\mu_3 \quad (A.58)$$

$$q_{8-1} = q_{21-13} = q_{20-14} = q_{30-26} = N_2\mu_2 + N_3\mu_3 \quad (A.59)$$

$$q_{21-1} = N_1\mu_1 + N_2\mu_2 + N_3\mu_3 \quad (A.60)$$

$$q_{32-19} = q_{33-20} = q_{34-22} = N_1\mu_1 P(n_1 = N_1 + 1) + N_2\mu_2 P(n_2 = N_2 + 1) \quad (A.61)$$

$$q_{28-16} = q_{31-20} = q_{34-24} = N_1\mu_1 P(n_1 = N_1 + 1) + N_3\mu_3 P(n_3 = N_3 + 1) \quad (A.62)$$

$$q_{12-7} = q_{25-20} = q_{34-30} = N_2\mu_2 P(n_2 = N_2 + 1) + N_3\mu_3 P(n_3 = N_3 + 1) \quad (A.63)$$

$$q_{34-20} = N_1\mu_1 P(n_1 = N_1 + 1) + N_2\mu_2 P(n_2 = N_2 + 1) + N_3\mu_3 P(n_3 = N_3 + 1) \quad (A.64)$$

$$q_{3-2} = q_{6-5} = q_{14-13} = N_4\mu_4 \quad (A.65)$$

$$q_{23-18} = N_2\mu_2 P(n_2 = N_2 + 1) + N_4\mu_4 \quad (A.66)$$

$$q_{9-8} = q_{17-15} = N_3\mu_3 P(n_3 = N_3 + 1) + N_4\mu_4 \quad (A.67)$$

$$q_{25-21} = N_2\mu_2 P(n_2 = N_2 + 1) + N_3\mu_3 P(n_3 = N_3 + 1) + N_4\mu_4 \quad (A.68)$$

$$q_{23-6} = q_{24-7} = q_{25-9} = N_1\mu_1 + N_2\mu_2 P(n_2 = N_2 + 1) \quad (A.69)$$

$$q_{17-3} = q_{22-7} = q_{25-11} = N_1\mu_1 + N_3\mu_3 P(n_3 = N_3 + 1) \quad (A.70)$$

$$q_{25-7} = N_1\mu_1 + N_2\mu_2 P(n_2 = N_2 + 1) + N_3\mu_3 P(n_3 = N_3 + 1) \quad (A.71)$$

$$q_{29-14} = q_{30-16} = q_{31-17} = N_1\mu_1 P(n_1 = N_1 + 1) + N_2\mu_2 \quad (A.72)$$

$$q_{9-3} = q_{22-16} = q_{31-27} = N_2\mu_2 + N_3\mu_3 P(n_3 = N_3 + 1) \quad (A.73)$$

$$q_{31-16} = N_1\mu_1 P(n_1 = N_1 + 1) + N_2\mu_2 + N_3\mu_3 P(n_3 = N_3 + 1) \quad (A.74)$$

$$q_{27-14} = q_{30-19} = q_{33-23} = N_1\mu_1 P(n_1 = N_1 + 1) + N_3\mu_3 \quad (A.75)$$

$$q_{11-6} = q_{24-19} = q_{33-29} = N_2\mu_2 P(n_2 = N_2 + 1) + N_3\mu_3 \quad (A.76)$$

$$q_{33-19} = N_1\mu_1 P(n_1 = N_1 + 1) + N_2\mu_2 P(n_2 = N_2 + 1) + N_3\mu_3 \quad (A.77)$$

$$q_{22-3} = N_1\mu_1 + N_2\mu_2 + N_3\mu_3 P(n_3 = N_3 + 1) \quad (A.78)$$

$$q_{24-6} = N_1\mu_1 + N_2\mu_2 P(n_2 = N_2 + 1) + N_3\mu_3 \quad (A.79)$$

$$q_{30-14} = N_1\mu_1 P(n_1 = N_1 + 1) + N_2\mu_2 + N_3\mu_3 \quad (A.80)$$

$$q_{16-2} = q_{19-5} = N_1\mu_1 + N_4\mu_4 \quad (A.81)$$

$$q_{7-2} = q_{19-13} = N_2\mu_2 + N_4\mu_4 \quad (A.82)$$

$$q_{7-5} = q_{16-13} = N_3\mu_3 + N_4\mu_4 \quad (A.83)$$

$$q_{20-2} = N_1\mu_1 + N_2\mu_2 + N_4\mu_4 \quad (A.84)$$

$$q_{20-5} = N_1\mu_1 + N_3\mu_3 + N_4\mu_4 \quad (A.85)$$

$$q_{20-13} = N_2\mu_2 + N_3\mu_3 + N_4\mu_4 \quad (A.86)$$

$$q_{22-8} = N_1\mu_1 + N_3\mu_3 P(n_3 = N_3 + 1) + N_4\mu_4 \quad (A.87)$$

$$q_{22-15} = N_2\mu_2 + N_3\mu_3 P(n_3 = N_3 + 1) + N_4\mu_4 \quad (A.88)$$

$$q_{24-18} = N_2\mu_2 P(n_2 = N_2 + 1) + N_3\mu_3 + N_4\mu_4 \quad (A.89)$$

### A. 3. Addendum

In this addendum to Appendix A, we present the description of the X's that are mentioned in Section 3.3.

$$X_1 = q_{1-2} + q_{1-5} + q_{1-8} + q_{1-13} + q_{1-15} + q_{1-18} + q_{1-21} \quad (A.90)$$

$$X_2 = P_2 q_{2-1} + P_5 q_{5-1} + P_8 q_{8-1} + P_{13} q_{13-1} + P_{15} q_{15-1} + P_{18} q_{18-1} + P_{21} q_{21-1} \quad (A.91)$$

$$X_3 = q_{2-1} + q_{2-3} + q_{2-7} + q_{2-8} + q_{2-15} + q_{2-16} + q_{2-20} + q_{2-21} \quad (A.92)$$

$$X_4 = P_1 q_{1-2} + P_3 q_{3-2} + P_7 q_{7-2} + P_8 q_{8-2} + P_{15} q_{15-2} + P_{16} q_{16-2} + P_{20} q_{20-2} + P_{21} q_{21-2} \quad (A.93)$$

$$X_5 = q_{3-2} + q_{3-4} + q_{3-7} + q_{3-9} + q_{3-16} + q_{3-17} + q_{3-20} + q_{3-22} \quad (A.94)$$

$$X_6 = P_2 q_{2-3} + P_4 q_{4-3} + P_7 q_{7-3} + P_9 q_{9-3} + P_{16} q_{16-3} + P_{17} q_{17-3} + P_{20} q_{20-3} + P_{22} q_{22-3} \quad (A.95)$$

$$X_7 = q_{4-3} + q_{4-9} + q_{4-17} + q_{4-22} \quad (A.96)$$

$$X_8 = P_3q_{3-4} + P_9q_{9-4} + P_{17}q_{17-4} + P_{22}q_{22-4} \quad (A.97)$$

$$X_9 = q_{5-1} + q_{5-6} + q_{5-7} + q_{5-8} + q_{5-18} + q_{5-19} + q_{5-20} + q_{5-21} \quad (A.98)$$

$$X_{10} = P_1q_{1-5} + P_6q_{6-5} + P_7q_{7-5} + P_8q_{8-5} + P_{18}q_{18-5} + P_{19}q_{19-5} + P_{20}q_{20-5} + P_{21}q_{21-5} \quad (A.99)$$

$$X_{11} = q_{6-5} + q_{6-7} + q_{6-10} + q_{6-11} + q_{6-19} + q_{6-20} + q_{6-23} + q_{6-24} \quad (A.100)$$

$$X_{12} = P_5q_{5-6} + P_7q_{7-6} + P_{10}q_{10-6} + P_{11}q_{11-6} + P_{19}q_{19-6} + P_{20}q_{20-6} + P_{23}q_{23-6} + P_{24}q_{24-6} \quad (A.101)$$

$$X_{13} = q_{7-2} + q_{7-3} + q_{7-5} + q_{7-6} + q_{7-9} + q_{7-11} + q_{7-12} + q_{7-20} + q_{7-22} + q_{7-24} + q_{7-25} \quad (A.102)$$

$$X_{14} = P_2q_{2-7} + P_3q_{3-7} + P_5q_{5-7} + P_6q_{6-7} + P_9q_{9-7} + P_{11}q_{11-7} + P_{12}q_{12-7} + P_{20}q_{20-7} + P_{22}q_{22-7} + P_{24}q_{24-7} + P_{25}q_{25-7} \quad (A.103)$$

$$X_{15} = q_{8-1} + q_{8-2} + q_{8-5} + q_{8-9} + q_{8-21} + q_{8-22} \quad (A.104)$$

$$X_{16} = P_1q_{1-8} + P_2q_{2-8} + P_5q_{5-8} + P_9q_{9-8} + P_{21}q_{21-8} + P_{22}q_{22-8} \quad (A.105)$$

$$X_{17} = q_{9-3} + q_{9-4} + q_{9-7} + q_{9-8} + q_{9-12} + q_{9-22} + q_{9-25} \quad (A.106)$$

$$X_{18} = P_3q_{3-9} + P_4q_{4-9} + P_7q_{7-9} + P_8q_{8-9} + P_{12}q_{12-9} + P_{22}q_{22-9} + P_{25}q_{25-9} \quad (A.107)$$

$$X_{19} = q_{10-6} + q_{10-11} + q_{10-23} + q_{10-24} \quad (A.108)$$

$$X_{20} = P_6q_{6-10} + P_{11}q_{11-10} + P_{23}q_{23-10} + P_{24}q_{24-10} \quad (A.109)$$

$$X_{21} = q_{11-6} + q_{11-7} + q_{11-10} + q_{11-12} + q_{11-24} + q_{11-25} \quad (A.110)$$

$$X_{22} = P_6q_{6-11} + P_7q_{7-11} + P_{10}q_{10-11} + P_{12}q_{12-11} + P_{24}q_{24-11} + P_{25}q_{25-11} \quad (A.111)$$

$$X_{23} = q_{12-7} + q_{12-9} + q_{12-11} + q_{12-25} \quad (A.112)$$

$$X_{24} = P_7q_{7-12} + P_9q_{9-12} + P_{11}q_{11-12} + P_{25}q_{25-12} \quad (A.113)$$

$$X_{25} = q_{13-1} + q_{13-14} + q_{13-15} + q_{13-16} + q_{13-18} + q_{13-19} + q_{13-20} + q_{13-21} \quad (A.114)$$

$$X_{26} = P_1q_{1-13} + P_{14}q_{14-13} + P_{15}q_{15-13} + P_{16}q_{16-13} + P_{18}q_{18-13} + P_{19}q_{19-13} + P_{20}q_{20-13} + P_{21}q_{21-13} \quad (A.115)$$

$$X_{27} = q_{14-13} + q_{14-16} + q_{14-19} + q_{14-20} + q_{14-26} + q_{14-27} + q_{14-29} + q_{14-30} \quad (A.116)$$

$$X_{28} = P_{13}q_{13-14} + P_{16}q_{16-14} + P_{19}q_{19-14} + P_{20}q_{20-14} + P_{26}q_{26-14} + P_{27}q_{27-14} + P_{29}q_{29-14} + P_{30}q_{30-14} \quad (A.117)$$

$$X_{29} = q_{15-1} + q_{15-2} + q_{15-13} + q_{15-17} + q_{15-21} + q_{15-22} \quad (A.118)$$

$$X_{30} = P_1q_{1-15} + P_2q_{2-15} + P_{13}q_{13-15}$$

$$+ P_{17}q_{17-15} + P_{21}q_{21-15} + P_{22}q_{22-15} \quad (A.119)$$

$$X_{31} = q_{16-2} + q_{16-3} + q_{16-13} + q_{16-14} + q_{16-17} + q_{16-20} + q_{16-22} + q_{16-27} + q_{16-28} + q_{16-30} + q_{16-31} \quad (A.120)$$

$$X_{32} = P_2q_{2-16} + P_3q_{3-16} + P_{13}q_{13-16} + P_{14}q_{14-16} + P_{17}q_{17-16} + P_{20}q_{20-16} + P_{22}q_{22-16} + P_{27}q_{27-16} + P_{28}q_{28-16} + P_{30}q_{30-16} + P_{31}q_{31-16} \quad (A.121)$$

$$X_{33} = q_{17-3} + q_{17-4} + q_{17-15} + q_{17-16} + q_{17-22} + q_{17-28} + q_{17-31} \quad (A.122)$$

$$X_{34} = P_3q_{3-17} + P_4q_{4-17} + P_{15}q_{15-17} + P_{16}q_{16-17} + P_{22}q_{22-17} + P_{28}q_{28-17} + P_{31}q_{31-17} \quad (A.123)$$

$$X_{35} = q_{18-1} + q_{18-5} + q_{18-13} + q_{18-21} + q_{18-23} + q_{18-24} \quad (A.124)$$

$$X_{36} = P_1q_{1-18} + P_5q_{5-18} + P_{13}q_{13-18} + P_{21}q_{21-18} + P_{23}q_{23-18} + P_{24}q_{24-18} \quad (A.125)$$

$$X_{37} = q_{19-5} + q_{19-6} + q_{19-13} + q_{19-14} + q_{19-20} + q_{19-23} + q_{19-24} + q_{19-29} + q_{19-30} + q_{19-32} + q_{19-33} \quad (A.126)$$

$$X_{38} = P_5q_{5-19} + P_6q_{6-19} + P_{13}q_{13-19} + P_{14}q_{14-19} + P_{20}q_{20-19} + P_{23}q_{23-19} + P_{24}q_{24-19} + P_{29}q_{29-19} + P_{30}q_{30-19} + P_{32}q_{32-19} + P_{33}q_{33-19} \quad (A.127)$$

$$X_{39} = q_{20-2} + q_{20-3} + q_{20-5} + q_{20-6} + q_{20-7} + q_{20-13} + q_{20-14} + q_{20-16} + q_{20-19} + q_{20-22} + q_{20-24} + q_{20-25} + q_{20-30} + q_{20-31} + q_{20-33} + q_{20-34} \quad (A.128)$$

$$X_{40} = P_2q_{2-20} + P_3q_{3-20} + P_5q_{5-20} + P_6q_{6-20} + P_7q_{7-20} + P_{13}q_{13-20} + P_{14}q_{14-20} + P_{16}q_{16-20} + P_{19}q_{19-20} + P_{22}q_{22-20} + P_{24}q_{24-20} + P_{25}q_{25-20} + P_{30}q_{30-20} + P_{31}q_{31-20} + P_{33}q_{33-20} + P_{34}q_{34-20} \quad (A.129)$$

$$X_{41} = q_{21-1} + q_{21-2} + q_{21-5} + q_{21-8} + q_{21-13} + q_{21-15} + q_{21-18} + q_{21-25} \quad (A.130)$$

$$X_{42} = P_1q_{1-21} + P_2q_{2-21} + P_5q_{5-21} + P_8q_{8-21} + P_{13}q_{13-21} + P_{15}q_{15-21} + P_{18}q_{18-21} + P_{25}q_{25-21} \quad (A.131)$$

$$X_{43} = q_{22-3} + q_{22-4} + q_{22-7} + q_{22-8} + q_{22-9} + q_{22-15} + q_{22-16} + q_{22-17} + q_{22-20} + q_{22-25} + q_{22-31} + q_{22-34} \quad (A.132)$$

$$X_{44} = P_3q_{3-22} + P_4q_{4-22} + P_7q_{7-22} + P_8q_{8-22} + P_9q_{9-22} + P_{15}q_{15-22} + P_{16}q_{16-22} + P_{17}q_{17-22} + P_{20}q_{20-22} + P_{25}q_{25-22} + P_{31}q_{31-22} + P_{34}q_{34-22} \quad (A.133)$$

$$X_{45} = q_{23-6} + q_{23-10} + q_{23-18} + q_{23-19} + q_{23-24} + q_{23-32} + q_{23-33} \quad (A.134)$$

$$X_{46} = P_6q_{6-23} + P_{10}q_{10-23} + P_{18}q_{18-23} + P_{19}q_{19-23} + P_{24}q_{24-23} + P_{32}q_{32-23} + P_{33}q_{33-23} \quad (A.135)$$

$$X_{47} = q_{24-6} + q_{24-7} + q_{24-10} + q_{24-11} + q_{24-18} + q_{24-19} + q_{24-20} + q_{24-23} + q_{24-25} + q_{24-33} + q_{24-34} \quad (A.136)$$

$$X_{48} = P_6q_{6-24} + P_7q_{7-24} + P_{10}q_{10-24} + P_{11}q_{11-24} + P_{18}q_{18-24} + P_{19}q_{19-24} + P_{20}q_{20-24}$$

$$+ P_{23}q_{23-24} + P_{25}q_{25-24} + P_{33}q_{33-24} + P_{34}q_{34-24} \quad (A.137)$$

$$X_{49} = q_{25-7} + q_{25-9} + q_{25-11} + q_{25-12} + q_{25-20} + q_{25-21} + q_{25-22} + q_{25-24} + q_{25-34} \quad (A.138)$$

$$X_{50} = P_7q_{7-25} + P_9q_{9-25} + P_{11}q_{11-25} + P_{12}q_{12-25} + P_{20}q_{20-25} + P_{21}q_{21-25} + P_{22}q_{22-25} + P_{24}q_{24-25} + P_{34}q_{34-25} \quad (A.139)$$

$$X_{51} = q_{26-14} + q_{26-27} + q_{26-29} + q_{26-30} \quad (A.140)$$

$$X_{52} = P_{14}q_{14-26} + P_{27}q_{27-26} + P_{29}q_{29-26} + P_{30}q_{30-26} \quad (A.141)$$

$$X_{53} = q_{27-14} + q_{27-16} + q_{27-26} + q_{27-28} + q_{27-30} + q_{27-31} \quad (A.142)$$

$$X_{54} = P_{14}q_{14-27} + P_{16}q_{16-27} + P_{26}q_{26-27} + P_{28}q_{28-27} + P_{30}q_{30-27} + P_{31}q_{31-27} \quad (A.143)$$

$$X_{55} = q_{28-16} + q_{28-17} + q_{28-27} + q_{28-31} \quad (A.144)$$

$$X_{56} = P_{16}q_{16-28} + P_{17}q_{17-28} + P_{27}q_{27-28} + P_{31}q_{31-28} \quad (A.145)$$

$$X_{57} = q_{29-14} + q_{29-19} + q_{29-26} + q_{29-30} + q_{29-32} + q_{29-33} \quad (A.146)$$

$$X_{58} = P_{14}q_{14-29} + P_{19}q_{19-29} + P_{26}q_{26-29} + P_{30}q_{30-29} + P_{32}q_{32-29} + P_{33}q_{33-29} \quad (A.147)$$

$$X_{59} = q_{30-14} + q_{30-16} + q_{30-19} + q_{30-20} + q_{30-26} + q_{30-27} + q_{30-29} + q_{30-31} + q_{30-33} + q_{30-34} \quad (A.148)$$

$$X_{60} = P_{14}q_{14-30} + P_{16}q_{16-30} + P_{19}q_{19-30} + P_{20}q_{20-30} + P_{26}q_{26-30} + P_{27}q_{27-30} + P_{29}q_{29-30} + P_{31}q_{31-30} + P_{33}q_{33-30} + P_{34}q_{34-30} \quad (A.149)$$

$$X_{61} = q_{31-16} + q_{31-17} + q_{31-20} + q_{31-22} + q_{31-27} + q_{31-28} + q_{31-30} + q_{31-34} \quad (A.150)$$

$$X_{62} = P_{16}q_{16-31} + P_{17}q_{17-31} + P_{20}q_{20-31} + P_{22}q_{22-31} + P_{27}q_{27-31} + P_{28}q_{28-31} + P_{30}q_{30-31} + P_{34}q_{34-31} \quad (A.151)$$

$$X_{63} = q_{32-19} + q_{32-23} + q_{32-29} + q_{32-33} \quad (A.152)$$

$$X_{64} = P_{19}q_{19-32} + P_{23}q_{23-32} + P_{29}q_{29-32} + P_{33}q_{33-32} \quad (A.153)$$

$$X_{65} = q_{33-19} + q_{33-20} + q_{33-23} + q_{33-24} + q_{33-29} + q_{33-30} + q_{33-32} + q_{33-34} \quad (A.154)$$

$$X_{66} = P_{19}q_{19-33} + P_{20}q_{20-33} + P_{23}q_{23-33} + P_{24}q_{24-33} + P_{29}q_{29-33} + P_{30}q_{30-33} + P_{32}q_{32-33} + P_{34}q_{34-33} \quad (A.155)$$

$$X_{67} = q_{34-20} + q_{34-22} + q_{34-24} + q_{34-25} + q_{34-30} + q_{34-31} + q_{34-33} \quad (A.156)$$

$$X_{68} = P_{20}q_{20-34} + P_{22}q_{22-34} + P_{24}q_{24-34} + P_{25}q_{25-34} + P_{30}q_{30-34} + P_{31}q_{31-34} + P_{33}q_{33-34} \quad (A.157)$$