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Chaotic logistic map sequences with good autocorrelation properties

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Abstract: Short *chaotic logistic map* (CLM) sequences with good autocorrelation properties are presented. The obtained sequences are synthesized by means of the logistic map function and are completely deterministic even though they appear to be random. The proposed short CLM sequences become a good alternative for synchronization purposes in digital communication systems. Also, the *peak sidelobe attenuation* (PSA) autocorrelation goodness metric is introduced. The PSA metric was employed as an objective function to find some of the best short CLM sequences with maximum correlation goodness.

Keywords: Autocorrelation goodness metric, chaotic logistic map sequences, frame synchronization, time synchronization

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1. Introduction

The logistic map is a mathematical function, originally derived for modeling the population size of a given ecosystem with constrictions, that may present a chaotic but deterministic behavioral. Its general expression is the following:

$$x(n+1) = x(n)[1 - x(n)]R \quad (1)$$

The properties of the logistic map have been widely studied (May, 1976) and it is well known that it generates a chaotic sequence when the *R* parameter is between 3.57 and 4. Furthermore, logistic function is very sensitive to the initial conditions: a minimal difference in the initial value, *x(0)*, produces a significant variation on the sequence values. Since chaotic sequences look like random sequences, the latter may be emulated by the former, with similar correlation properties but with the advantage, for the chaotic sequences, of being generated in a deterministic and simple way.

Sequences with good correlation properties has been studied, for many decades, since are very appreciated in many fields and are especially important for synchronization purposes in the digital communications field (Kocabas & Atalar, 2003). Currently, mathematically defined sequences, such as the Zadoff-Chu sequences, are widely used for frame synchronization purposes and their properties have been thoroughly analyzed (Kang et al., 2012). Even, bio inspired heuristics have been proposed to synthesize sequences with good autocorrelation properties (Militzer et al., 1998).

In this work, the chaotic logistic map (CLM) sequences are proposed as an alternative to generate sequences with good autocorrelation properties, by means of maximizing the proposed peak sidelobe attenuation metric (PSA), used as an objective function, over the search space defined by parameters *R* and *x(0)*. Although the application of the autocorrelation properties of chaotic sequences for synchronization and communications purposes is not a new idea (Rao & Howard, 1996), the main contribution of this work is to provide practical guidelines to determinate the optimum parameters for the synthesis of chaotic sequences with good correlation properties.

The rest of this work is presented as follows: In Section 2, the proposed peak sidelobe attenuation (PSA) metric is introduced; in Section 3, the used experimental methodology is specified; In Section 4, related to the maximization of the proposed autocorrelation goodness function for logistic sequences, the experimental results are presented, and the explored search space is described; finally, in Section 5, the conclusions are drawn.

2. The proposed autocorrelation goodness metric

The use of specific sequences with good autocorrelation properties is usual for time synchronization in digital communication systems. Specifically, within the OFDM systems, frame synchronization is achieved taking advantage of cyclic prefixes (CP), originally added to avoid inter symbol interference (ISI). In these kinds of synchronization problems, the natural solution is the aperiodic autocorrelation computation to find the position of the cyclic prefix and, thus, the beginning of the OFDM frame. Additionally, if the frame synchronization sequence is known, it also may be used for channel estimation purposes.

Some cross correlation and autocorrelation metrics are typically computed for the sequences focusing on their periodic or aperiodic performance. For example, let *x = (x(0), x(1),…, x(N-1))* denote a sequence with *N* samples, where $x(k) \in \mathbb{R}$, the periodic, *c(k)*, and aperiodic, *r(k)*, autocorrelations for the sequence are defined as:

$$c(k) = \sum_{n=0}^{N-1} x(n)x(n+k)_{mod\ N}, 0 \le k \le (N-1) \quad (2)$$

$$r(k) = \sum_{n=0}^{N-1} x(n)x(n+k), 0 \le k \le (N-1) \quad (3)$$

In theory, an infinite sequence of additive white gaussian noise (AWGN) is a sequence with an ideal autocorrelation function. That is, an autocorrelation function that is nonzero for all nonzero values of the *k* offset (see Fig. 1). For a sufficiently large AWGN sequence, the autocorrelation function tends to show the profile of the ideal autocorrelation function, but with non-zero values for most of the non-zero *k*, with the consequent appearance of sidelobes (see Fig. 2). In principle, the longer the length of the AWGN noise sequence, the closer the approximation to the profile of the ideal autocorrelation function and, therefore, the distance, *d*, between the maximum autocorrelation value and the maximum of the sidelobe peaks will also be maximum (see Fig. 3).

The sidelobe problem increases as the length of the sequence decreases. But this is not only true for AWGN sequences, but for any sequence that is designed to exhibit an autocorrelation profile like that of an ideal autocorrelation function. This is the main challenge for designing short sequences with good autocorrelation properties.

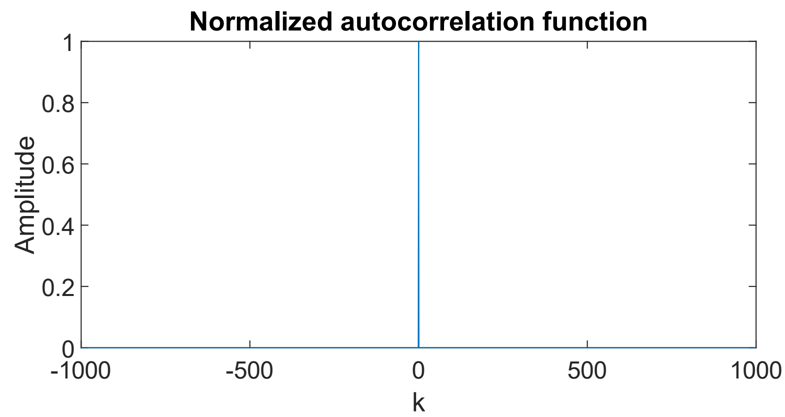

Figure 1. Ideal profile for autocorrelation function.

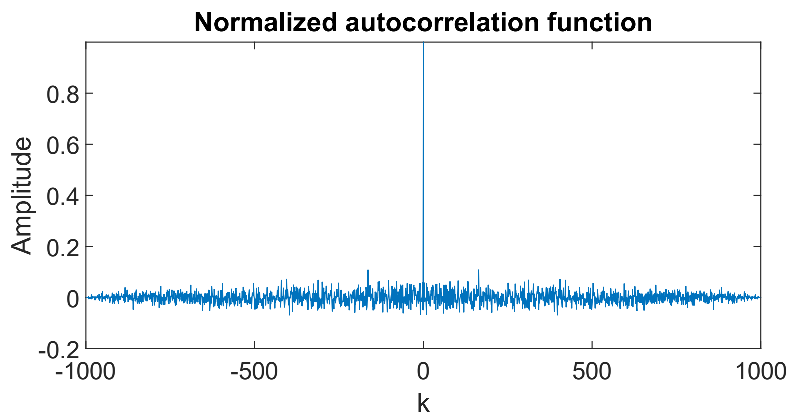

Figure 2. Profile for autocorrelation function of a real AWGN sequence with 1000 values.

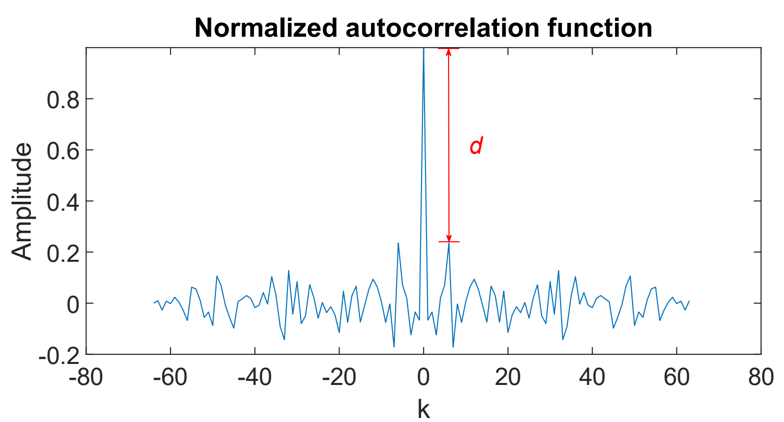

Figure 3. Profile for autocorrelation function of a real AWGN sequence with 64 values.

To evaluate the autocorrelation performance of a given sequence, it is common to apply the following three metrics (Soltanalian & Stoica, 2012):

-The Peak Sidelobe Level

$$PSL = max\{r(k)\}_{k=1}^{N-1}, \quad (4)$$

- the Integrated Sidelobe Level

$$ISL = \sum_{k=1}^{N-1} r(k)^2, \quad (5)$$

- and the Merit Factor

$$MF = \frac{r(0)^2}{2\sum_{k=1}^{N-1} r(k)^2} = \frac{r(0)^2}{2 \times ISL}. \quad (6)$$

In this work, the *peak sidelobe attenuation* is introduced:

$$PSA = \frac{r(0)}{PSL}. \quad (7)$$

This improved metric allows us to maximize the ratio for maximum autocorrelation value with respect to the peak sidelobe level, to maximize the synchronization performance of a sequence, and this with only a slight increase in computational cost over the PSL metric.

The proposed PSA metric provides a quantitative tool to maximize the distance *d*. Unlike the PSL metric, which yields an absolute value, the PSA metric results in a relative metric that can be expressed in dB in order to consider the distance *d*, in terms of the equivalent attenuation value that represents the peak value of the largest sidelobe with respect to the maximum value of the autocorrelation function. The computational cost implied by this improvement is minimal, with respect to that of the PSL metric, as can be seen in Table 1, where the computational costs of each metric are expressed in terms of the operations of multiplication and accumulation (*MAC*); subtraction and comparison (*SNC*); and division (*DIV*).

Table 1. Computational cost for the reference autocorrelation metrics.

Metric	Cost
PSL	$(N-1) \times SNC$
ISL	$(N-1) \times MAC$
MF	$(N+1) \times MAC + DIV$
PSA	$(N-1) \times SNC + DIV$

In Table 1, taking into account that *MAC* and *DIV* operations are normally much more expensive than an *SNC* operation, it can be seen that the order of magnitude of the computational cost of the PSL and PSA metrics is practically the same between them. On the other hand, the ISL and MF metrics also have a similar order of magnitude between them, but significantly higher than the PSL and PSA metrics. In our work, due to the brute-force algorithm used, it is necessary to have a metric for the performance of the autocorrelation function that allows maximizing the distance *d* with the lowest possible computational cost.

3. Experimental methodology

In this work, a theoretical approach was not used but a completely empirical and experimental one. The aim is to obtain short sequences with good correlation properties, for possible application to concrete engineering problems. The value of the theoretical approach is not underestimated, but the practical usefulness of concrete functional and realizable sequences is very appreciated, even if these are obtained by a brute-force computational algorithm.

As already mentioned above, the proposed peak sidelobe attenuation (PSA) is chosen as objective function to explore its search space to maximize the autocorrelation goodness for the chaotic sequences generated by Eq. 1. Specifically, for chaotic sequences generated by means of the logistic map function, the useful search space is defined by the *R* parameter range within (3.57, 4.0] and *x(0)* range within (0.0, 1.0]. To outline the general shape of the objective function, the experimental methodology is supported by a brute-force algorithm. Basically, the algorithm executes the following steps, according to a specific resolution for both *R* and *x(0)* ranges:

Data: Resolution and range for R and x(0) parameters.
Result: maximum PSA value.
While not processed all pairs {R, x (0)} **do**
    Generate a logistic map sequence with the required size.
    Compute the autocorrelation sequence.
    Compute and store the PSA metric.
Plot all the computed PSA values for the search space;
Print the maximum PSA value on the search space and their related parameters;

4. Experimental results and search space description

The experimental results are computationally quite easy to reproduce. The search space is defined by both the parameters of the logistic map function and the resolution used. For demonstration purposes, two resolutions will be used to outline the overall PSA function surface over the search space: 1) search space with 50×50 points of resolution, and 2) search space with 400×400 points of resolution. The resulting PSA function surfaces are plotted in Fig. 4 (3D view) and Fig. 5 (2D view), for a resolution of 50×50 points, and in Fig. 6 (3D view) and Fig. 7 (2D view), for a resolution of 400×400 points.

In Figures 4 to 7, it must be remarked that the objective function surface shows a corrugated shape with the highest PSA local maximums in the band defined by *R*>3.96. Also, it was experimentally observed that the reached maximum PSA value increases as the search space resolution increases. For example, after synthesizing CLM sequences of 64 values for each coordinate *(R, x(0))* on the search space with a 50×50 low resolution, such as depicted in Fig. 4 and Fig. 5, result in local maximums PSA values around 8.68 dB; while in the case of a 400×400 resolution, as depicted in Fig. 6 and Fig. 7, result in local maximum PSA values around 10.0 dB. Experimentally it was possible to prove that PSA reaches values around 11.0 dB on search spaces with 100,000×100,000 points resolution. Since the search algorithm used in this work is based on a brute-force tactic, these optimization searches have a relatively long-time executions, but this issue may be improved by using a heuristic optimization technique.

In Table 2, PSA local maximums are reported for some of the best chaotic logistic map sequences, with 64 values, over a search space with resolution of 100,000×100,000 points. The initial condition, *x(0)*, of the logistic map, for the synthesis of the sequences, is shown in the 3rd column, just next to the value for the *R* parameter.

Table 2. PSA metrics and logistic map function parameters for some of the best short CLM sequences (with a length of 64 values), found from the local PSA maximums over search spaces with resolution of 100,000×100,000 points.

PSA metric [dB]	*R* value	*x (0)* value
11.183	3.99119	0.06952
11.097	3.99894	0.17857
11.046	3.99986	0.20092
10.968	3.99881	0.36061
10.886	3.99913	0.47951
10.898	3.99926	0.53799
11.538	3.99871	0.67709
11.005	3.99099	0.71311
11.097	3.99894	0.82143
11.113	3.99807	0.90204

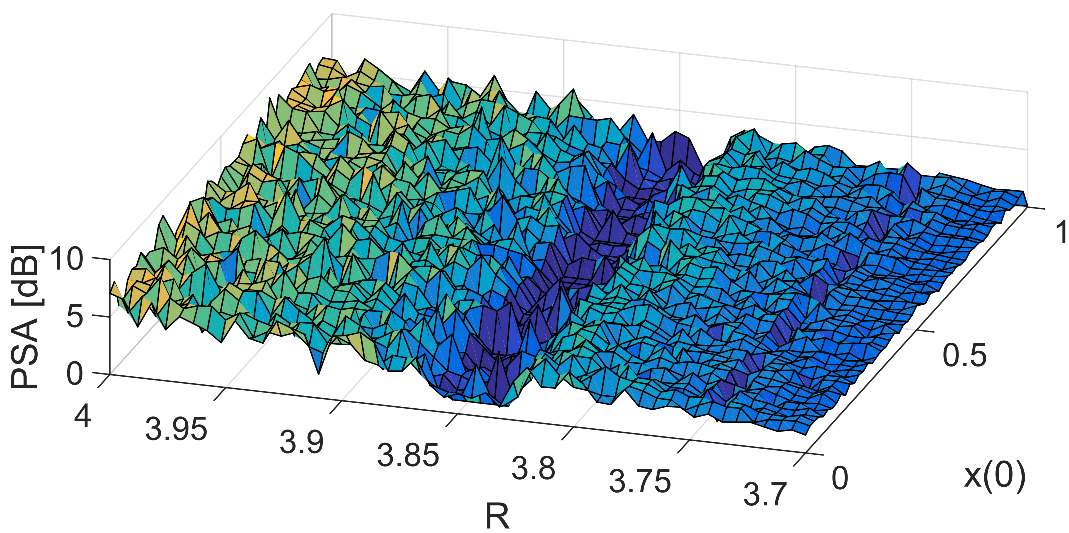

Figure 4. 3D view of PSA function surface for search space with 50x50 points.

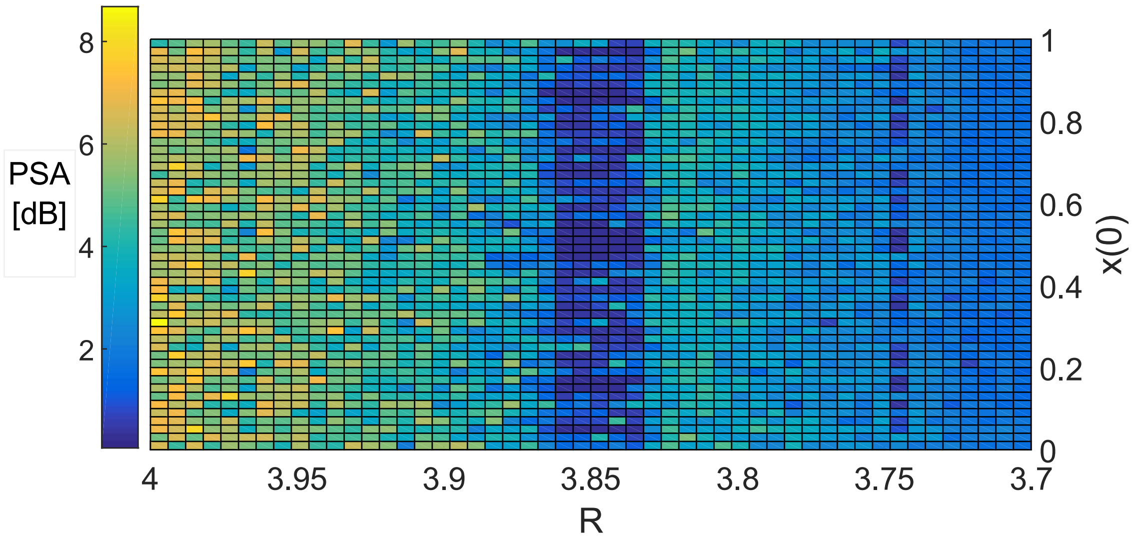

Figure 5. 2D view of PSA function surface for search space with 50x50 points.

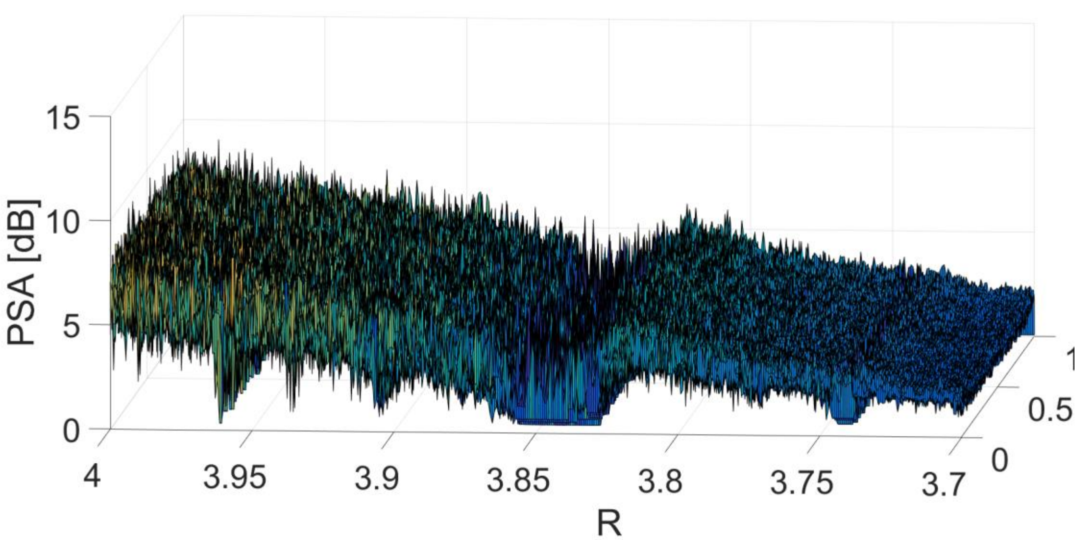

Figure 6. 3D view of PSA function surface for search space with 400x400 points.

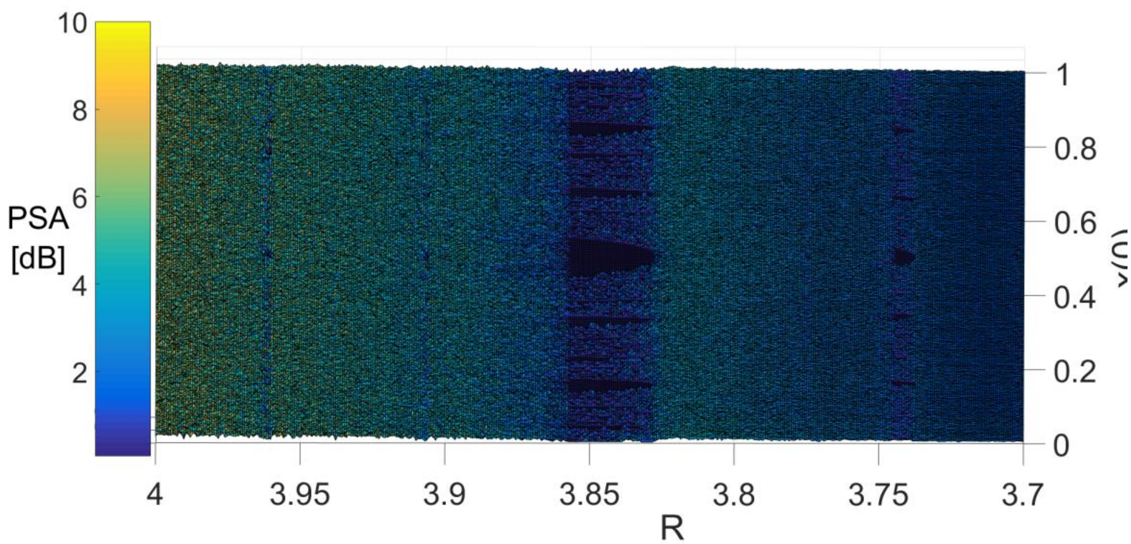

Figure 7. 2D view of PSA function surface for search space with 400x400 points.

As a benchmark, the PSA metrics for some of the best short Zadoff-Chu sequences (with a length of 64 values) are reported in Table 3. It must be remembered that the Zadoff-Chu sequences, for an even number of values, is defined as (Chu, 1972):

$$z(n) = \exp\left(i\frac{M\pi n^2}{N}\right) \tag{8}$$

where N is the even size and M is an integer relatively prime to N.

Table 3. PSA metrics and M parameter for some of the best short Zadoff-Chu sequences (with a length of 64 values).

PSA metric [dB]	*M* value
7.2312	5
6.8756	7
7.4162	9
7.9959	13
8.6597	21
8.2284	35
9.4423	43
8.3289	51
7.9377	55
8.7702	63

It is important to note that, for any short sequence, it is difficult to reach a high value for any autocorrelation goodness metric. That is the main reason to choose sequences with 64 values of length to compare the autocorrelation performance of the chaotic logistic map sequences with respect to the well-known Zadoff-Chu sequences. Specifically, the LTE standard specifies the use of short sequences, with only 63 values, to get fine timing by means a cross-correlation with the Primary Synchronization Signal (PSS) [i.e., see (Donarski et al., 2014) and the references therein].

After comparing the autocorrelation PSA goodness metrics, reported in Tables 2 and 3, it is clear the superiority of the chaotic logistic map (CLM) sequences, with respect to Zadoff-Chu sequences. In Fig. 8, the overall performance for PSA autocorrelation metric for both CLM and Zadoff-Chu sequences is depicted, and it can be remarked the average 3dB PSA gain of CLM sequences over the reference Zadoff-Chu ones. These results show that, for same length of 64 values, the proposed CLM sequences produce autocorrelations goodness twice higher than Zadoff-Chu ones, which is very convenient to improve any time or frame synchronization process. Finally, considering that both sequences are fully deterministic and synthesized by mathematical expressions, the chaotic logistic map sequences become an interesting alternative to produce sequences with good autocorrelation properties.

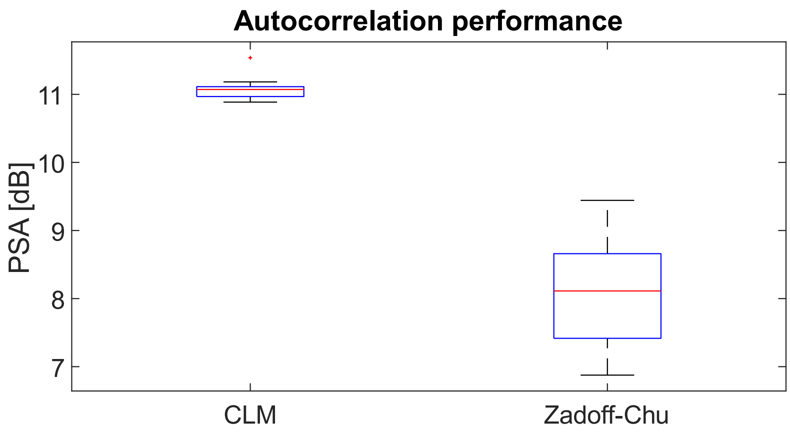

Figure 8. Autocorrelation performance for both CLM and Zadoff-Chu sequences with a length of 64 values.

5. Conclusions

The logistic map function was used to synthesize some short chaotic sequences that show an autocorrelation goodness gain of 3dB over the bets Zadoff-Chu sequences with the same length. To better measure the autocorrelation goodness for a given sequence, the peak sidelobe attenuation (PSA) metric has been introduced. After empirical exploration of the related search space, trying to maximize the PSA metric, it was observed that the objective function surface clearly shows a recognizable shape that allows the optimization algorithms to find local maximums with the best autocorrelation goodness. For demonstration purposes, after explore the search space with resolution of 0.00001 for the logistic map function parameters, *R* and *x(0)*, the maximum autocorrelation goodness reached by the best chaotic logistic map (CLM) sequences (with a length of 64 values) showed a significant improvement (twice in average) with respect to the autocorrelation goodness for the best Zadoff-Chu sequences with the same length, which represents an evident advantage that allow us to successfully apply them for both time and frame synchronization purposes.

Although these results have been obtained empirically, by means of a brute-force experimental strategy, the potential of the logistic map function to generate short chaotic sequences with excellent autocorrelation properties is remarkable and promising. As future work, a deeper theoretical study is pending to try to formally explain this interesting and useful property that may emerge in this type of chaotic sequence.

Conflict of interest

The authors have no conflict of interest to declare.

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