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Some novel fixed-point theorems in Hausdorff spaces

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Abstract: In this paper existence and uniqueness of fixed points are proved for self maps, satisfying a new contraction without assuming the compatibility and commutative property of maps. Some remarks and applications to integral type contraction are given to illustrate the importance of our results. An open problem for future research is also given.

Keywords: mappings, fixed point, Hausdorff space, integral contraction, applications

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1. Introduction

In contemporary topological theory, iterative techniques are broadly used to find roots of linear and nonlinear systems of equations, differential equations and integral equations. Banach (1922) introduced a well-liked iterative method. Several authors have extended, improved and generalized Banach's theorem in different ways (Alamgir et al., 2020; Bondar, 2011; Gupta et al., 2015; Gupta et al., 2020; Gupta & Verma, 2020; Jaggi,1976; Samet, & Yazidi, 2011).

Popa (1983) generalized the result of Banach through Hausdorff topological spaces and proved some unique fixed-point theorems.

Theorem 1.1 (Popa, 1983) - Let $F: X \to X$ be a continuous mapping of a Hausdorff space X and let $H: X \times X \to [0, +\infty)$ be continuous mapping so that, for each $x \neq y \in X$,

I. $H(x,y) \neq 0$
II. $H(Fx,Fy) \leq \alpha\left(\frac{H(x,Fx)H(y,Fy)}{H(x,y)}\right) + \beta\big(H(x,y)\big)$
III. $H^2(x,y) \geq H(x,x)H(y,y)$,

where $\alpha, \beta > 0$ and $\alpha + \beta < 1$. If for some $x_0 \in X$, the sequence $x_n = \{F^n x_0\}$ has a convergent subsequence, then F has a unique fixed point.

Remark 1.1: - Any metric space is a Hausdorff-metric space, or easily, Hausdorff spaces in the induced topology.

Jungck (1976) proved a common fixed-point theorem for commuting maps so that one of them is continuous. Sessa (1982) generalized the concept of commuting maps to weakly commutating pairs of self-mappings. Furthermore, Jungck (1986) generalized this idea; first, to compatible mappings and then to weakly compatible mappings (Jungck, 1996).

Results by Banach further extended in several directions for self and pairs of mappings. Some of the latest results on fixed- and common-fixed points can be found in (Gupta & Verma, 2020; Shahi et al., 2014).

Branciari (2002) introduced a new definition for the Lebesgue-integrable function and proved a fixed-point theorem satisfying the contractive condition of an integral type as an analog of the Banach contraction principal.

Definition 1.1 (Branciari, 2002) - A function defined as $\Phi(t) = \varphi: [0, +\infty\,) \to [0, +\infty\,)$, is Lebesgue-summable for each compact of R_+. Let us define its permittivity $A: [0, +\infty) \to [0, +\infty)$ as $A(t) = \int_0^t \varphi\,(t)dt, \quad t > 0$ is well defined, non-decreasing and continuous. Moreover, if for each $\epsilon > 0, A(\epsilon) > 0$; this permittivity fulfills $A(t) = 0$ if and only if $t = 0$. Then, it is called a Lebesgue-integrable function.

Branciari (2002) result was further studied by many other authors and lot of generalizations have been done (Gupta et al., 2012; Gupta & Mani, 2013) and the references there in

Samet and Yazidi (2011) gave an extension of Branciari (2002) result by using rational inequality in Hausdorff topological spaces and proved the following theorem:

Theorem 1.2 (Samet & Yazidi, 2011)- Let X be a Hausdorff space and $H: X \times X \to [0, +\infty)$ be a continuous mapping so that

$$H(x,y) \neq 0, \quad \forall \quad x,y \in X \quad and \quad x \neq y.$$

Let F be self maps of X satisfying the contractive condition so that for each $x \neq y \in X$:

$$\int_0^{H(Fx,Fy)} \varphi\,(t)dt \leq \alpha \int_0^{M(x,y)} \varphi\,(t)dt + \beta \int_0^{H(x,y)} \varphi\,(t)dt,$$

where $M(x,y) = \frac{H(y,Fy)[1+H(x,Fx)]}{[1+H(x,y)]}$,

$\alpha, \beta > 0$ are constants with $\alpha + \beta < 1$ and $\varphi(t)$ is a Lebesgue-integrable function. If for some $x_0 \in X$, the sequence of iterates $\{F^n x_0\}$ has a subsequence $\{F^{n_k} x_0\}$ converging to $z \in X$; then z is a fixed point of F.

Our main results are the following theorems.

2. Main Results

Theorem 2.1:- Let $F: X \to X$ be a continuous mapping in Hausdorff space X and $H: X \times X \to [0, +\infty)$ be a continuous mapping so that for each $x \neq y \in X$

$$H(x,y) \neq 0 \tag{1}$$
$$H(Fx,Fy) \leq \alpha M(x,y) + \beta H(x,y), \tag{2}$$

Where

$$M(x,y) = max\left\{H(x,y), \frac{H(x,Fx)H(y,Fy)}{H(x,y)}\right\}, \tag{3}$$

$\alpha, \beta > 0$ are constants with $\alpha + \beta < 1$. If for some $x_0 \in X$, the sequence $x_n = \{F^n x_0\}$ has a subsequence $x_{n_k} = \{F^{n_k} x_0\}$ converging to $z \in X$; then, z is a fixed point of F.

Proof:- Let us choose $x_0 \in X$ so that $Fx_0 = x_1$. Now let us define a sequence $\{x_n\}$ in X so that $Fx_n = x_{n+1}$.

First, let us suppose that there exists $m \in N$ so that $F^m x_0 = F^{m+1} x_0$; then, for all $n \geq m$, we get $F^n x_0 = F^m x_0$ and $z = F^m x_0$ is a fixed point of F.

Second, assume that $F^n x_0 \neq F^{n+1} x_0$, for $n \in N$.; then, from Eq. 2, we have:

$$\begin{aligned} H(x_n, x_{n+1}) &= H(Fx_{n-1}, Fx_n) \\ &\leq \alpha M(x_{n-1}, x_n) + \beta H(x_{n-1}, x_n), \end{aligned} \tag{4}$$

where, from Eq. 3:

$$M(x_{n-1},x_n) = max\left\{H(x_{n-1},x_n),\frac{H(x_{n-1},Fx_{n-1})H(x_n,F_{xn})}{H(x_{n-1},x_n)}\right\} = max\{H(x_{n-1},x_n),H(x_n,x_{n+1})\}. \quad (5)$$

Let us suppose $H(x_n,x_{n+1}) > H(x_{n-1},x_n)$; then by using Eq. 4,

$$H(x_n,x_{n+1}) \le \alpha H(x_n,x_{n+1}) + \beta H(x_{n-1},x_n) \le \frac{\beta}{1-\alpha} H(x_{n-1},x_n) < H(x_{n-1},x_n). \quad (6)$$

Also, if $H(x_n,x_{n+1}) \le H(x_{n-1},x_n)$; then, again from Eq. 4:

$$H(x_n,x_{n+1}) \le (\alpha+\beta)H(x_{n-1},x_n) < H(x_{n-1},x_n). \quad (7)$$

With the use of Eq. 6 and Eq. 7, and by repeating the above process up to n times, we get:

$$H(x_n,x_{n+1}) < H(x_{n-1},x_n) <\cdots< H(x_1,x_2) < H(x_0,x_1).$$

Thus, we obtain a monotone sequence of non-negative real numbers, which must converge with all its subsequence to some real no $u \in X$.

Next, let us claim that z is a fixed point of F.

To prove this, let us suppose z is not a fixed point of F.

The continuity of F and Himplies:

$$\begin{aligned} H(z,Fz) &= H\left[\lim_{k\to\infty} x_{n_k}, F\left(\lim_{k\to\infty} x_{n_k}\right)\right] \\ &= H\left[\lim_{k\to\infty} x_{n_k}, \lim_{k\to\infty} x_{n_k+1}\right] \\ &= \lim_{k\to\infty} H\left[x_{n_k}, x_{n_k+1}\right] = u \\ &= \lim_{k\to\infty} H\left[x_{n_k+1}, x_{n_k+2}\right] \\ &= H\left[\lim_{k\to\infty} x_{n_k+1}, \lim_{k\to\infty} x_{n_k+2}\right] \\ &= H\left[F\lim_{k\to\infty} x_{n_k}, F^2\left(\lim_{k\to\infty} x_{n_k}\right)\right] \\ &= H(Fz,F^2z) \\ &\le \alpha M(z,Fz) + \beta H(z,Fz) \end{aligned} \quad (8)$$

On using Eq. 3:

$$M(z,Fz) = max\left\{H(z,Fz),\frac{H(z,Fz)H(Fz,F^2z)}{H(z,Fz)}\right\} = max\{H(z,Fz),H(Fz,F^2z)\} = max\{H(z,Fz),H(z,Fz)\}.$$

Hence, from Eq. 8, we get :

$$H(z,Fz) \le (\alpha+\beta)H(z,Fz).$$

This is a contradiction to our assumption; thus, $Fz = z$. That is, $z$is a fixed point of F. This completes the proof of Theorem 2.1.

In our next result, we introduce a new contraction to establish a common fixed-point theorem for a pair of self maps in Hausdorff spaces without using the compatibility and commutative property.

Theorem 2. 2:- Let $F,G: X \to X$ are continuous mappings in Hausdorff space X and $H: X \times X \to [0,+\infty)$ be a continuous mapping so that for each $x \neq y \in X$

$$H(x,y) \neq 0; \quad (9)$$
$$H(Fx,Gy) \le \alpha M(x,y) + \beta H(x,y) \quad (10)$$

where

$$M(x,y) = max\left\{H(x,y),\frac{H(x,Fx)H(y,Gy)}{H(x,y)}\right\} \quad (11)$$

$\alpha,\beta > 0$ are constants with $\alpha+\beta < 1$. If for some $x_0 \in X$, sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to $z \in X$; then, z is a common fixed point of maps F and G.

Proof:- Le us choose $x_0 \in X$ so that $Fx_0 = x_1$ and $Gx_1 = x_2$. Now, let us construct a sequence $\{x_n\}$ in X so that $Fx_{2n} = x_{2n+1}$ and $Gx_{2n+1} = x_{2n+2}$ for $n = 0,1,2,\cdots$

First, let us suppose that there exists $m \in N$ so that$F^m x_0 = F^{m+1}x_0$; then, for all $n \ge m$, we get $F^n x_0 = F^m x_0$ and $z = F^m x_0$ is a fixed point of F.

Second, let us assume that $F^n x_0 \neq F^{n+1}x_0$, n ∊ N, then from Eq. 10, we have:

$$H(x_{2n+1},x_{2n+2}) = H(Fx_{2n},Gx_{2n+1}) \le \alpha M(x_{2n},x_{2n+1}) + \beta H(x_{2n},x_{2n+1}), \quad (12)$$

Where, from Eq. 11, we have:

$$\begin{aligned} M(x_{2n},x_{2n+1}) &= max\left\{H(x_{2n},x_{2n+1}),\frac{H(x_{2n},Fx_{2n})H(x_{2n+1},Gx_{2n+1})}{H(x_{2n},x_{2n+1})}\right\} \\ &= max\left\{H(x_{2n},x_{2n+1}),\frac{H(x_{2n},x_{2n+1})H(x_{2n+1},x_{2n+2})}{H(x_{2n},x_{2n+1})}\right\} \\ &= max\{H(x_{2n},x_{2n+1}),H(x_{2n+1},x_{2n+2})\} \end{aligned} \quad (13)$$

Let us suppose that $H(x_{2n+1},x_{2n+2}) > H(x_{2n},x_{2n+1})$; then, from Eq. 12:

$$H(x_{2n+1},x_{2n+2}) \le \alpha H(x_{2n+1},x_{2n+2}) + \beta H(x_{2n+1},x_{2n+2}) \le \frac{\beta}{1-\alpha} H(x_{2n},x_{2n+1}) < H(x_{2n},x_{2n+1}). \quad (14)$$

Also, if $H(x_{2n+1},x_{2n+2}) \le H(x_{2n},x_{2n+1})$; again, from Eq. 12:

$$H(x_{2n+1}, x_{2n+2}) \leq (\alpha+\beta)H(x_{2n}, x_{2n+1}) < H(x_{2n}, x_{2n+1}).$$

Repeating the above process n times, we get

$$H(x_{2n+1}, x_{2n+2}) < H(x_{2n}, x_{2n+1}) < \cdots < H(x_1, x_2) < H(x_0, x_1).$$

Thus we get a monotone sequence $\{x_n\}$ of non-negative real numbers, which must converge with all its subsequence to some real no $u \in X$.

Now, we show that z is fixed point of F and G. First, we show that z is fixed point of F.

Let us suppose$Fz \neq z$.

Let us consider sequence $\{x_n\}$ has a subsequence $\{x_{2n_k}\}$ that converges to some real number z; then, from the continuity of F, G and H, we have:

$$\begin{aligned} H(z, Fz) &= H\left[\lim_{k\to\infty} x_{2n_k}, F\left(\lim_{k\to\infty} x_{2n_k}\right)\right] \\ &= H\left[\lim_{k\to\infty} x_{2n_k}, \lim_{k\to\infty} x_{2n_k+1}\right] = u \\ &= \lim_{k\to\infty} H\left[x_{2n_k+1}, x_{2n_k+2}\right] \\ &= H\left[\lim_{k\to\infty} x_{2n_k+1}, \lim_{k\to\infty} x_{2n_k+2}\right] \\ &= H\left[F\lim_{k\to\infty} x_{2n_k}, GF\left(\lim_{k\to\infty} x_{2n_k}\right)\right] \\ &= H(Fz, GFz) \\ &\leq \alpha M(z, Fz) + \beta H(z, Fz), \end{aligned} \tag{15}$$

Where, from Eq. 11:

$$M(z, Fz) = max\{H(z, Fz), H(Fz, GFz)\} < H(z, Fz). \tag{16}$$

Thus, from Eq. 15 and Eq. 16:

$$H(z, Fz) \leq \alpha H(z, Fz) + \beta H(z, Fz) = (\alpha+\beta)H(z, Fz) < H(z, Fz)$$

This is a contradiction. Thus, z is a fixed point of F. Analogously, we can show that z is fixed point of G. This completes the proof of Theorem 2.1.

In order to get the uniqueness of the fixed point for the maps (in Theorem 2.1 and Theorem 2.2), we consider the following assumption:

$$H(x, x)H(y, y) \leq H^2(x, y) \tag{17}$$

Theorem 2.3: If we add condition (17) to the hypothesis of Theorem 2.1, we get a unique fixed point of map F.

Proof:- We have proved that $Fz = z$. Let us suppose there exists another point $w \in X$ so that $Fw = w$.

From Eq 2:

$$H(z, w) = H(Fz, Fw) \leq \alpha M(z, w) + \beta H(z, w), \tag{18}$$

Where,

$$\begin{aligned} M(z, w) &= max\left\{H(z, w), \frac{H(z, Fz)H(w, Fw)}{H(z, w)}\right\} \\ &= max\left\{H(z, w), \frac{H(z,z)H(w,w)}{H(z,w)}\right\} \\ &\leq max\{H(z, w), H(z, w)\} = H(z, w). \end{aligned} \tag{19}$$

From Eq 18 and Eq 19, we get a contradiction.

Thus z is a unique fixed point of F

Theorem 2.4:- If we add condition (17) to the hypothesis of Theorem 2.2, we get a unique common fixed point for maps F and G.

Proof:- We have proved that $Fz = z$ and $Gz = z$. Let us suppose there exists another point $w \in X$, so that $Fw = w$ and $Gw = w$:

From Eq. 10:

$$H(z, w) = H(Fz, Gw) \leq \alpha M(z, w) + \beta H(z, w), \tag{20}$$

On using (17), we have:

$$\begin{aligned} M(z, w) &= max\left\{H(z, w), \frac{H(z, Fz)H(w, Gw)}{H(z, w)}\right\} \\ &= max\left\{H(z, w), \frac{H(z,z)H(w,w)}{H(z,w)}\right\} \leq H(z, w) \end{aligned} \tag{21}$$

From Eq. 20 and Eq 21, we get a contradiction. Thus, z is a unique common fixed point of $F$and$G$.

Remark 1:- Note that in the above theorems (Theorem 2.1 and Theorem 2.2), the continuity of maps is necessary to get the fixed point; otherwise, the fixed point cannot be guaranteed

Remark 2:- Authors leave here an open problem for further research to get the uniqueness of fixed points in Theorem 2.1 and Theorem 2.2 without assuming condition (17).

3. Applications for the integral type contraction

In this section, we discuss the existence and uniqueness of the fixed point for integral type contractive mappings. Besides being a proper extension, results obtained here are weaker than the result obtained by Samet and Yazidi (2011), and some other existing results.

Theorem 3.1: Let $F: X \to X$ be a continuous mapping in Hausdorff space X and $H: X \times X \to [0, +\infty)$ be a continuous mapping so that for each $x \neq y \in X$:

$H(x,y) \neq 0;$

$$\int_0^{H(Fx,Fy)} \varphi\,(t)dt \leq \alpha \int_0^{M(x,y)} \varphi\,(t)dt + \beta \int_0^{H(x,y)} \varphi\,(t)dt,$$

where

$$M(x,y) = max\left\{H(x,y), \frac{H(x,Fx)H(y,Fy)}{H(x,y)}\right\},$$

$\alpha, \beta > 0$ are constants with $\alpha + \beta < 1$ and $\varphi(t)$ is a Lebesgue-integrable function. If for some $x_0 \in X$, the sequence $x_n = \{F^n x_0\}$ has a subsequence $x_{n_k} = \{F^{n_k} x_0\}$ converging to $z \in X$; then z is a fixed point F.

Proof:- By assuming $\varphi(t) = 1$ and using Theorem 2.1, we obtain the desired result.

Theorem 3.2: Let $F, G: X \to X$ are continuous mappings in Hausdorff space X and $H: X \times X \to [0, +\infty)$ be a continuous mapping so that for each $x \neq y \in X$:

$H(x,y) \neq 0;$

$$\int_0^{H(Fx,Gy)} \varphi\,(t)dt \leq \alpha \int_0^{M(x,y)} \varphi\,(t)dt + \beta \int_0^{H(x,y)} \varphi\,(t)dt,$$

where

$$M(x,y) = max\left\{H(x,y), \frac{H(x,Fx)H(y,Gy)}{H(x,y)}\right\},$$

$\alpha, \beta > 0$ are constants with $\alpha + \beta < 1$ and $\varphi(t)$ is a Lebesgue-integrable function. If for some $x_0 \in X$, the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to $z \in X$, then z is a common fixed point of maps F and G.

Proof:- By taking $\varphi(t) = 1$ in Theorem 2.2, we get the result.

4. Conclusions

In this paper, firstly, we derived a fixed-point result (Theorem 2.1) for a self map. In Theorem 2.2, we introduced a contraction to get a common fixed point for a pair of self maps without using the compatibility and commutative property of maps. Theorem 2.3 and Theorem 2.4 proved the uniqueness of the fixed point by assuming an additional assumption on the maps. Some observational remarks and an open problem are given for further research.

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