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Banach contraction theorem on fuzzy cone b-metric space

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Abstract: In the present paper the notion of fuzzy cone b -metric space has been introduced. Here we have defined fuzzy cone b -contractive mapping, and Banach contraction theorem for single mapping and pair of mappings has been proved in the setting of fuzzy cone b -metric space.

Keywords: Fixed point, fuzzy metric space, fuzzy cone metric space, fuzzy cone b -metric space.

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1. Introduction

Maurice Frechet introduced the metric space in 1906. Since then it has been generalized by many researchers. The notion of b metric was introduced by Bakhtin (1989) which was further used by Czerwik (1993, 1998) to prove many results.

Definition 1.1. (Czerwik, 1993) Let $\boldsymbol{M}$ be a non empty set and $\lambda \geq 1$ be a given real number. A function $\boldsymbol{d}: M \times M \rightarrow [0, \infty)$ is a $\boldsymbol{b}$-metric on $\boldsymbol{M}$ if the following conditions hold for all $x, y, z \in M$,

(B_1) $d(x,y) = 0 iff x = y$;
(B_2) $d(x,y) = d(x,y)$;
(B_3) $d(x,z) \leq \lambda[d(x,y) + d(y,z)]$;

The triplet (M, d, λ) is called b-metric space.

Examples and fixed point theorems related to b metric space are mentioned in (Ansari, Gupta, & Mani 2020; Boriceanu, 2009; Boriceanu Bota, & Petrusel 2010; Shatanawi, Pitea, & Lazovic, 2014).

The concept of b-metric space is broader than concept of metric space, when we take $\lambda = 1$ in b-metric space then we get metric space.

Huang and Zhang (2007) generalized the concept of metric space by introducing the concept of cone metric space. In the research by Huang and Zhang (2007) real numbers are replaced with an ordered Banach space and some fixed point theorems for non linear mappings are proved. After the work of Huang and Zhang (2007), lot of literature appeared related to the study of cone metric spaces. Details are available (Jankovi'c, Kadelburg, & Radenovi'c, 2011; Latif, Hussain, & Ahmad, 2016; Mehmood, Azam, & Ko'cinac, 2015; Shatanawi, Karapinar, & Aydi,2012).

Zadeh (1965) introduced the concept of fuzzy set theory. After his work many researchers started applying this new concept to classical theories. In particular, Kramosil and Michalek (1975) introduced the new concept fuzzy metric space and proved many results. George and Veeramani (1994) introduced a stronger form of fuzzy metric space. Afterwards, many mathematicians studied fixed point theorems in the related spaces (Chauhan & Utreja, 2013; Chauhan & Kant, 2015; Gupta, Saini, & Verma, 2020; Gupta, & Verma, 2020). Czerwik (1998) introduced b metric space and proved some results. The concept of b metric space is the extension to metric space.

Hussain and Shah (2011) introduced the concept of cone b metric space, which generalizes both b-metric space and cone metric space.

Oner, Kandemire, and Tanay (2015) applied the concept of fuzziness to cone metric space and introduced fuzzy cone metric space as a generalized form of fuzzy metric space given by George and Veeramani (1994). They proved some basic properties and fixed point theorems under this space. We can see related work in (Oner, 2016a, 2016b; Priyobarta, Rohen, & Upadhyay, 2016).

In this paper, we have introduced the concept of fuzzy cone b metric space in the sense of George and Veeramani (1994). Here we combine the notion of cone b metric space with the concept of fuzziness in the sense of George and Veeramani (1994) and proved new version of Banach contraction principle using this concept. We have defined fuzzy cone b contractive mapping and proved the fuzzy cone b-Banach contraction theorem for single mappind as well as the pair of mappings. Other important results which are helpful in this study are (Abbas, Khan & Radenovic 2010; Ali & Kanna 2017: Boriceanu, Bota & Petrusel 2010: Li & Jiang 2014; Turkoglu & Abuloha 2010).

Some more basic definitions which are used directly or indirectly are mentioned below:

Definition 1.2. (Schweizer & Sklar, 1960) The binary operation $*: [0,1] \times [0,1] \rightarrow [0,1]$ is called continuous t-norm if $*$ satisfies the following conditions for al $a, b, c, d \in [0,1]$,

1. $*$ is commutative and associative;
2. $*$ is continuous;
3. $a * 1 = a, \forall a \in [0,1]$;
4. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$;

Example 1.1. Some examples of continuous t-norms are $\wedge$, and $*L$, which are defined by $c \wedge d = min\{c, d\}, c.d = cd$ (usual multiplication in [0,1]) and $c * Ld = max\{c + d - 1, 0\}$.

Definition 1.3. (George & Veeramani, 1994) The triple $(Y, N, *)$ is said to be fuzzy metric space if Y is an arbitrary set, $*$ is a continuous t-norm and N is a fuzzy set on $Y \times Y \times (0, \infty)$ such that for all $a, b, c \in Y$ and $s, t > 0$, we have

1. $N(a,b,t) > 0$;
2. $N(a,b,t) = 1$ if and ony if $a = b$;
3. $N(a,b,t) = N(b,a,t)$;
4. $N(a,c,t+s) \geq N(a,b,t) * N(b,c,s)$;
5. $N(a,b,.): (0,\infty) \rightarrow [0,1]$ is continuous.

Definition 1.4. (Sedghi & Shobe, 2012) Let Y be a non empty set, $*$ a continuous t-norm and let $k \geq 1$ be a given real number. A fuzzy set N in $Y \times Y \times (0,\infty)$ is called b-fuzzy metric if for any $a, b, c \in Y$, and $t, s > 0$, the following conditions hold:

1. $N(a,b,0) > 0$;
2. $N(a,b,t) = 1$ if and only if $a = b$;
3. $N(a,b,t) = N(b,a,t)$;
4. $N(a,c,t+s) \geq N(a,b,\frac{t}{k}) * N(b,c,\frac{s}{k})$;

5. $N(a,b,.): (0,\infty) \to [0,1]$ is continuous.

Throughout this paper B denotes a real a Banach space and θ denotes the zero of B.

Definition 1.5. (Huang & Zhang, 2007) Let Q be the subset of B. Then Q is called a cone if

1. Q is closed, non empty, and $Q \neq \{\theta\}$;
2. if $c,d \in [0,\infty)$ and $u,v \in Q$, then $cu+dv \in Q$;
3. if both $u \in Q$ and $-u \in Q$, then $u=\theta$.

For a given cone $Q \subset B$ a partial ordering $\prec$ on B via Q is defined by $u \prec v$ if and only if $v-u \in Q$. $u \prec v$ stands for $u < v$ and $u \neq v$, while $u << v$ stands for $v-u \in int(Q)$, where $int(Q)$ is the set of all interior points of Q. In this paper, we assume that all cones has non empty interior.

Definition 1.6. (Oner et al., 2015) A three tuple $(Y,N,*)$ is said to be a fuzzy cone metric space if Q is a cone of B, Y is an arbitrary set, $*$ is a continuous t-norm and N is a fuzzy set on $Y^2 \times int(Q)$ satisfying the following conditions for $a,b,c \in Y$ and $t,s \in int(P)$,

1. $N(a,b,t) > 0$ and $N(a,b,t)=1$ iff $a=b$;
2. $N(a,b,t)=N(b,a,t)$;
3. $N(a,c,t+s) \geq N(a,b,t) * N(b,c,s)$;
4. $N(a,b,.): int(Q) \to [0,1]$ is continuous.

Definition 1.7. (Oner et al., 2015) Consider a fuzzy cone metric space $(Y,N,*)$, $y \in Y$ and $\{y_n\}$ be a sequence in Y, then

1. $\{y_n\}$ is said to be converge to y if for $t >> \theta$ and $\alpha \in (0,1)$ there exists natural number n_1 such that $N(y_n,y,t) > 1-\alpha$ for all $n > n_1$.

We denote it by $\lim_{n\to\infty} y_n = y$ or $y_n \to y$ as $n \to \infty$;

2. $\{y_n\}$ is said to be a Cauchy sequence if for $\alpha \in (0,1)$ and $t >> \theta$ there exists natural number n_1 such that $N(y_m,y_n,t) > 1-\alpha$ for all $m,n \geq n_1$;
3. $(Y,N,*)$ is said to be a complete cone metric space if every Cauchy sequence is convergent in Y;
4. $\{y_n\}$ is said to be fuzzy cone contractive if there exists $\alpha \in (0,1)$ such that

$$\frac{1}{N(y_{n+1},y_{n+2},t)} - 1 \leq \alpha\left(\frac{1}{N(y_n,y_{n+1},t)} - 1\right) \quad \text{for all} \quad t \geq \theta, \quad n \in N.$$

2. Main results

Definition 2.1 Let Y be a non empty arbitrary set, $*$ is a continuous t-norm, N is a fuzzy set on $Y \times Y \times Int(Q)$, Q is a cone of $\boldsymbol{B}$ (Real Banach space). A quadruple $(Y,N,*,\lambda)$ is said to be fuzzy cone b-metric space if following conditions are satisfied for all $a,b,c \in Y$ and $t,s \in IntQ, \lambda \geq 1$,

FCNB1: $N(a,b,t) > 0; N(a,b,0)=0$;
FCNB2: $N(a,b,t) = 1 for all t > 0 iff a = b$
FCNB3: $N(a,b,t) = N(b,a,t)$
FCNB4: $N(a,b,t) * N(b,c,s) \leq N(a,c,\lambda(t+s)), s \geq 0$
FCNB5: $N(a,b,.): int(Q) \to [0,1]$ is continuous and $\lim_{n\to\infty} N(a,b,t) = 1$.

Example 2.1: Let $B = R^2$. Then $Q = \{(r_1,r_2): r_1,r_2 \geq 0\}$ subset of B, is a normal cone with normal constant $k=1$.
Let $Y = R$, $a*b = ab$ and $M: X^2 \times int(Q) \to [0,1]$, defined by
$M(x,y,t) = \frac{1}{e^{\frac{|x-y|}{\|\lambda t\|}}}$ for all $x,y \in X$ and $t \geq 0$.

FCNB1: $M(x,y,0) = \frac{1}{e^{\frac{|x-y|}{0}}} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0;$

$$M(x,y,t) > 0 if t > 0.$$

FCNB2: $M(x,y,t) = 1$ for all $t > 0$ iff $x=y$. i.e. $M(x,y,t) = \frac{1}{e^{\frac{|x-y|}{\|\lambda t\|}}} = \frac{1}{e^0} = 1.$

FCNB3: $M(x,y,t) = \frac{1}{e^{\frac{|x-y|}{\|\lambda t\|}}} = \frac{1}{e^{\frac{|y-x|}{\|\lambda t\|}}} = M(y,x,t).$

FCNB4: $s \leq t+s \leq \lambda(t+s)$, and $t \leq t+s \leq \lambda(t+s)$, as $\lambda \geq 1$.

This gives, $\|s\| \leq \|\lambda(t+s)\|$ and $\|t\| \leq \|\lambda(t+s)\|$, we have $\frac{\|\lambda(t+s)\|}{\|s\|} \geq 1$ and $\frac{\|\lambda(t+s)\|}{\|t\|} \geq 1$.

Now, $|x-z| \leq |x-y| + |y-z|$,
we can write, $|x-z| \leq |x-y|\frac{\|\lambda(t+s)\|}{\|t\|} + |y-z|\frac{\|\lambda(t+s)\|}{\|s\|}$,
this implies, $\frac{|x-z|}{\|\lambda(t+s)\|} \leq \frac{|x-y|}{\|t\|} + \frac{|y-z|}{\|s\|}$,
so, one can get $e^{\frac{|x-z|}{\|\lambda(t+s)\|}} \leq e^{\frac{|x-y|}{\|t\|}} + e^{\frac{|y-z|}{\|s\|}}$ or $\frac{1}{e^{\frac{|x-z|}{\|\lambda(t+s)\|}}} \geq \frac{1}{e^{\frac{|x-y|}{\|t\|}}} + \frac{1}{e^{\frac{|y-z|}{\|s\|}}}$.

Thus, $M(x,z,\lambda(t+s)) \geq M(x,y,t) * M(y,z,s)$.

FCNB5: Define $f_1: IntQ \to (0,\infty)$ such that $f_1(t) = \|t\| = \sqrt{r_1^2 + r_2^2}$ and $f_2: (0,\infty) \to [0,1]$, $f_2(v) = e^{\frac{|x-y|}{v}}$.

Then $M(x,y,.): IntQ \to [0,1]$ is composite function of f_1 and f_2.
Both f_1 and f_2 are continuous, hence $M(x,y,.)$ is also continuous and $\lim_{t\to\infty} M(x,y,t) = \lim_{t\to\infty} \frac{1}{e^{\frac{|x-y|}{\|t\|}}} = 1.$

Definition 2.2: Let $(Y,N,*,\lambda)$ be a fuzzy cone b-metric space. $R: Y \to Y$ be a self mapping. Then R is said to be fuzzy cone b-contractive if there exist $\alpha \in (0,1)$ such that

$\frac{1}{N(Rx,Ry,t)} - 1 \leq \alpha\left(\frac{1}{N(x,y,t)} - 1\right)$ for $x, y \in Y$ and $t \geq \theta$, where θ denotes the zero of B and α is known as contraction constant of R.

Definition 2.3: Let $(Y, M, *, \lambda)$ be a fuzzy cone b -metric space and $\{x_n\}$ be a sequence in Y. Then $\{x_n\}$ is said to be fuzzy cone b -contractive if
$\frac{1}{M(x_{n+1},x_{n+2},t)} - 1 \leq \alpha \left(\frac{1}{M(x_n,x_{n+1},t)} - 1\right)$ for all $t \geq \theta$, $n$is natural number,$\alpha \in (0,1)$.

Definition 2.4: Let $(Y, M, *, \lambda)$ be a fuzzy cone b -metric space.$A, B: Y \to Y$ are self mappings. Then mappings $A$and$B$are known as fuzzy cone b -contractive if$\alpha \in (0,1)$ such that
$\frac{1}{N(Ax,By,t)} - 1 \leq \alpha \left(\frac{1}{N(x,y,t)} - 1\right)$ for $t \geq \theta$, α is called contraction constant of $A$and B.

Fuzzy Cone b -Banach Contraction Theorem

Theorem 2.1: Let $(X, M,^*, \lambda)$be complete fuzzy cone b -metric space in which fuzzy cone b -contractive sequence is Cauchy and$S, T \to R$ be fuzzy cone contractive mappings and $S(X) \subseteq T(X)$. Then S and Thave unique common fixed point.

Proof: Let $x_0 \in X$, define a sequence $\{x_n\}$ such that $x_{2n+1} = Sx_{2n}, x_{2n+2} = Tx_{2n+1}$ for $n = 0,1,2...$

First we show that sub sequence $\{x_{2n}\}$ is a Cauchy sequence

$$\frac{1}{M(x_{2n+1}, x_{2n+2}, t)} - 1 = \frac{1}{M(Sx_{2n}, Tx_{2n+1}, t)} - 1$$
$$\leq \alpha\left(\frac{1}{M(x_{2n}, x_{2n+1}, t)} - 1\right)$$
$$= \alpha\left(\frac{1}{M(Sx_{2n-1}, Tx_{2n}, t)} - 1\right) \leq \alpha^2\left(\frac{1}{M(x_{2n-1}, x_{2n}, t)} - 1\right),$$

Continue in this way, we get

$$\frac{1}{M(x_{2n+1}, x_{2n+2}, t)} - 1 \leq \alpha^{2n+1} \frac{1}{M(x_o, x_1, t)} - 1.$$

Then x_{2n} is a Cauchy sequence in $X$and $X$is complete .

Therefore x_{2n} converges to y for some $y \in X$. Then using Theorem 2.10 (Nadaban, 2016)
we have,
$\frac{1}{M(Sx_{2n},Tx_{2n+1},t)} - 1 \leq \alpha\left(\frac{1}{M(x_{2n+1},x_{2n+2},t)} - 1\right) \leq \alpha\left(\frac{1}{M(y,y,t)} - 1\right)$

this gives, $\frac{1}{M(Sx_{2n},Tx_{2n+1},t)} = 1$. Thus, $Sx_{2n} = Tx_{2n+1}$ and therefore $Sy = Ty$ as $n \to \infty$.

Hence y is a coincidence point of $S$and T.

Now we will prove that $y$is a fixed point of $S$and T.

Since, $\frac{1}{M(Sy,Tx_{2n},t)} - 1 \leq \alpha\left(\frac{1}{M(y,x_{2n},t)} - 1\right)$,

this gives, $\frac{1}{M(Sy,x_{2n+1},t)} - 1 \leq \alpha\left(\frac{1}{M(y,y,t)} - 1\right)$ as $x_{2n} \to y$ and $Tx_{2n} = x_{2n+1}$,
thus $\frac{1}{M(Sy,y,t)} - 1 \leq 0$. Hence $Sy = y$.

For uniqueness, let $u$is also a fixed point of $S$and T, ie. $Su = Tu = u$.

Therefore, $\frac{1}{M(y,u,t)} - 1 = \frac{1}{M(Sy,Tu,t)} - 1 \leq \alpha\left(\frac{1}{M(y,u,t)} - 1\right)$
ie. $(1 - \alpha)\left(\frac{1}{M(y,u,t)} - 1\right) \leq 0$, this gives,$M(y, u, t) = 1$.

Thus $y = u$. and hence$y$is a unique common fixed point of $S$and T.

Corollary 2.1 (Fuzzy Cone b -Banach Contraction Theorem) Let $(X, M, * \lambda)$ be a complete fuzzy cone b -metric space in which fuzzy cone b -contractive sequence is Cauchy and $S: X \to X$be a fuzzy cone contractive mapping. Then S has unique fixed point.

Proof. If we put $S = T$in Theorem 2.1, we get the result.

Definition 2.5: Let $(X, M, * \lambda)$ be fuzzy cone b -metric space. A self mapping Ton$X$ is called Chauhan-Gupta contraction if it satisfies the following condition for all$x, y \in X, k \geq 0$ and $a, b \in 0,1)$suchthat$a + b < 1, a < 1 - k$,

$$\frac{1}{M(Tx, Ty, t)} - 1 \leq a\left(\frac{1}{M(x, Tx, t) * M(y, Tx, t)} - 1\right) + b\left(\frac{1}{M(y, Ty, t)} - 1\right) + k\left(\frac{1}{\min\{M(x,Ty,t), M(y,Tx,t)\}} - 1\right). \tag{1}$$

Theorem 2.2: Let(X, M, t, λ) be a complete fuzzy cone b -metric space. Let $T: X \to X$ is a Chauhan-Gupta contraction given by (1). Then$T$has a unique fixed point in$X$.

Proof: Let $x_0 \in X$ and define a sequence $\{x_n\}$ by $x_n = Tx_{n-1}$ for $n \geq 0$. Then by (1), for $t \geq 0, n \geq 1$,

$$\frac{1}{M(Tx, Ty, t)} - 1 \leq a\left(\frac{1}{M(x, Tx, t) * M(y, Tx, t)} - 1\right) + b\left(\frac{1}{M(y, Ty, t)} - 1\right) + k\left(\frac{1}{\min\{M(x, Ty, t), M(y, Tx, t)\}} - 1\right).$$

this gives,

$$\frac{1}{M(x_n,x_{n+1},t)} - 1 = \frac{1}{M(Tx_{n-1},Tx_n,t)} - 1$$
$$\leq a\left(\frac{1}{M(x_{n-1},Tx_{n-1},t)*M(x_n,Tx_{n-1},t)} - 1\right) + b\left(\frac{1}{M(x_n,Tx_n,t)} - 1\right) + k\left(\frac{1}{min\{M(x_{n-1},Tx_n,t),M(x_n,Tx_n,t)\}} - 1\right),$$
$$\leq a\left(\frac{1}{M(x_{n-1},x_n,t)*M(x_n,x_n,t)} - 1\right) + b\left(\frac{1}{M(x_n,x_{n+1},t)} - 1\right)$$

$$+k\left(\frac{1}{min\{M(x_{n-1},x_{n+1},t),M(x_n,x_n,t)\}}-1\right)$$
$$\le a\left(\frac{1}{M(x_{n-1},x_n,t)}-1\right)+b\left(\frac{1}{M(x_n,x_{n+1},t)}-1\right)+k(0),$$

we get, $\left(\frac{1}{M(x_n,x_{n+1},t)}-1\right) \quad \le \quad \frac{a}{(1-b)}\left(\frac{1}{M(x_{n-1},x_n,t)}-1\right)$

This implies, $\frac{1}{M(x_n,x_{n+1},t)}-1\le h\left(\frac{1}{M(x_{n-1},x_n,t)}-1\right)\le h^n\left(\frac{1}{M(x_0,x_1,t)}-1\right)$, where $h=\frac{a}{(1-b)}<1$ as $a+b<1$. $\{x_n\}$ is a fuzzy cone b -contractive sequence and therfore one can get $\lim_{n\to\infty}M(x_n,x_{n+1},t)=1$ for $t\ge 0$.

Now, for $m>n\ge n_0$,

$$\frac{1}{M(x_n,x_m,t)}-1\le\left(\frac{1}{M(x_n,x_{n+1},t)}-1\right)+\left(\frac{1}{M(x_{n+1},x_{n+2},t)}-1\right)+\ldots+\left(\frac{1}{M(x_{m-1},x_m,t)}-1\right)$$
$$\le h^n\left(\frac{1}{M(x_0,x_1,t)}-1\right)+h^{n+1}\left(\frac{1}{M(x_0,x_1,t)}-1\right)+\ldots+h^{m-1}\left(\frac{1}{M(x_0,x_1,t)}-1\right)$$
$$=(h^n+h^{n+1}+\ldots+h^{m-1})\left(\frac{1}{M(x_0,x_1,t)}-1\right)\to 0$$

as $n\to\infty$.

Thus $\lim_{n\to\infty}M(x_n,x_m,t)=1$, which gives $\{x_n\}$ is a Cauchy sequence. The completenes of X, one can say $\lim_{n\to\infty}x_n=u$.

Now, $\frac{1}{M(x_{n+1},Tu,t)}-1=\frac{1}{M(Tx_n,Tu,t)}-1$

$$\le a\left(\frac{1}{M(x_n,Tx_n,t)*M(u,Tx_n,t)}-1\right)+b\left(\frac{1}{M(u,Tu,t)}-1\right)$$
$$+k\left(\frac{1}{min\{M(x_n,Tu,t),M(u,Tx_n,t)\}}-1\right)$$
$$=a\left(\frac{1}{M(x_n,x_{n+1},t)*M(u,x_{n+1},t)}-1\right)+b\left(\frac{1}{M(u,Tu,t)}-1\right)$$
$$+k\left(\frac{1}{min\{M(x_n,Tu,t),M(u,x_{n+1},t)\}}-1\right)$$
$$=a\left(\frac{1}{M(u,Tu,t)*M(u,u,t)}-1\right)+b\left(\frac{1}{M(u,Tu,t)}-1\right)$$
$$+k\left(\frac{1}{min\{M(u,Tu,t),M(u,u,t)\}}-1\right),$$

we get, $\frac{1}{M(u,u,t)}-1\le b\left(\frac{1}{M(u,Tu,t)}-1\right)$ for $t\ge 0$ and $b<1$.

Hence u is a fixed point of T.

For uniqueness, let v is another fixed point of T.

$$\frac{1}{M(u,v,t)}-1=\frac{1}{M(Tu,Tv,t)}-1$$
$$\le a\left(\frac{1}{M(u,Tu,t)*M(v,Tu,t)}-1\right)+b\left(\frac{1}{M(v,Tv,t)}-1\right)$$
$$+k\left(\frac{1}{min\{M(u,Tv,t),M(v,Tu,t)\}}-1\right)$$
$$=a\left(\frac{1}{M(u,u,t)*M(v,u,t)}-1\right)+b\left(\frac{1}{M(v,v,t)}-1\right)$$
$$+k\left(\frac{1}{min\{M(u,v,t),M(v,u,t)\}}-1\right),$$

implies, $\frac{1}{M(u,v,t)}-1\le(a+k)\left(\frac{1}{M(v,u,t)}-1\right)$, where $a+k<1$.

Thus $M(u,v,t)=1$ and this gives $u=v$.

Hence, u is a unique fixed point of T.

3. Conclusions

In this article, we Introduced the idea of fuzzy cone b -metric space and the fuzzy cone b –contractive mapping has been defined. Also, the Banach contraction theorem has been proved in the setting of fuzzy cone b metric space. Based on the results in this paper, interesting researches may be prospective. In the future study, one can establish the integral version of fixed point theorem in this space and can also think of establishing some new fixed point results in fuzzy cone b -metric space. The work presented here is likely to provide a ground to the researchers to do work in different structures by using these conditions.

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