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Discrete time multi-server loss systems and stochastic approximation with delayed groups of customers

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Abstract: The stochastic approximation procedure with delayed groups of delayed customers is investigated. The Robbins-Monro stochastic approximation procedure is adjusted to be usable in the presence of delayed groups of delayed customers. Two loss systems are introduced to get an accurate description of the proposed procedure. Each customer comes after fixed time intervals with the stage of the following customer is accurate according to the outcome of the preceding one, where the serving time of a customer is assumed to be a discrete random variable. Some applications of the procedure are given where the analysis of their results is obtained. The most important result is that efficiencies of the procedure are increased by increasing the service-time distributions as well as service times of customers. This new situation can be applied to increase the number of served customers where the number of served groups will also be increased. The results obtained seem to be acceptable. In general, our proposal can be utilized for other stochastic approximation procedures to increase the production in many fields such as medicine, computer sciences, industry, and applied sciences.

Keywords: Analysis of results, customers, groups, efficiency, loss systems, modified procedure, service time, stochastic approximation

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1. Introduction

The Robbins-Monro Stochastic approximation procedure Robbins and Monro (1951) is an iterative algorithm for finding the root of an equation, or the solution of a system of equations, where it cannot be computed directly, but only estimated with the estimation subject to random error.

The Robbins-Monro stochastic approximation procedure with delayed observations already utilized for a geometrical delay distribution Dupač and Herkenrath (1985) by allocating experiments into $\boldsymbol{K}$ parallel series, and constructing a global approximation by averaging the $\boldsymbol{K}$ individual ones. The time loss caused by delayed observations, or its complement, the performance of the procedure was studied in the quoted paper. There is a large typecast and a lot of papers on the topic (Blum, 1954; Cheung & Elkind, 2010; Combes, 2013; Jonckheere & Leskelä, 2007; Joseph et al., 2007; Leroux, 1992; Mahmoud, 1988; Nevel´son & Has′minskiĭ, 1972), we shall use the review papers (Mahmoud & Rasha, 2005; Mahmoud & Atwa, 2011; Mahmoud et al., 2015; Mahmoud et al., 2017) as references. In stochastic procedures, customers follow each other after fixed time-intervals with the point of the next customer is corrected according to the result of the preceding one. In our previous work Mahmoud and Atwa, (2011), we applied the Robbins-Monro Stochastic approximation procedure in the existence of compound delayed observations, where the random time delay distribution of compound observations was estimated to get the approximated efficiency of the procedure. Recently, in the paper Mahmoud et al. (2017) we applied the Robbins-Monro procedure in the presence of groups of delayed observations by investigating two loss systems. A sever of one of the two loss systems cans serve a group of delayed observations, where the number of served observations could be increased.

Here we modified the Robbins-Monro stochastic approximation procedure to be usable in existence of delayed groups of delayed multiservice customers and investigating two loss systems as a new application of the Robbins-Monro procedure. The two loss systems can be described as follows:

A group of delayed customers arrives to the service system each time unit where the service time is an integer-valued random variable, servers are parallel, and there is no waiting places if all servers are busy. According to this new way, a server of the second loss system cans serve a group, where the first $(r-1)$ customers are served without delay and the r^{th} one is delayed or all the r customers are served without delay. If the j^{th} customer is delayed, then all its preceding customers are served without delays, and all its next customers will be lost. As a result of applying the two loss systems to the proposed procedure, the number of served customers can be increased, if the number of served customers without delay is maximum and this lead to increase the performance (efficiency) of the procedure. This approach is not applied in the review paper Mahmoud et al. (2017), for in this paper the problem was examined by designing two specific loss systems where a server cannot receive more than one observation of the group while our approach is more general, for the server cans receive a group of customers where this will increase the number of served customers. In the papers (Mahmoud & Atwa, 2011; Mahmoud et al., 2015; Mahmoud et al., 2017; Nevel´son & Has′minskiĭ, 1972), the problem was examined by applying some specific loss systems where servers cannot extradite (serve) any observation during the time between any two sequential arrivals.

However, in the proposed procedure, we introduced a new situation by investigating two loss systems, where the server cans receive more than one customer during the time between any two consecutive groups. In fact, the investigation of the aforesaid two loss systems was made exactly by this application, where it will minimize the number of missing customers.

Consequently, the performance (efficiency) of the procedure is increased and this ability used as an application in industry to increase the production of items of some industrial projects. The service time of a group equals the sum of service times of the customers where it is independent of the number of customers and this can be used in increasing the number of served customers during the same service time of the group.

The probability service time of a group equals the sum of products of probabilities service time of customers without delay terminated by with or without delay. If the probabilities service time of customers are without delay and terminated with retard, then all following customers are lost. The investigated procedure is new and we foresee that it can be used to any stochastic approximation or recursive estimation procedure.

2. Constructions of the first and second loss systems

The first loss system is constructed as follows:

Consider the service system $GI_g/GI_g/K/0$ where symbols have the following meaning respectively, arrivals and service times are positive integers; K the number of servers that are parallel; no queues. Assume that the distribution of arrival times is deterministic, that is a group of customers arrives each time unit. Groups are lost if all servers are busy, in this case the service system is called a loss system with delayed groups of customers and is denoted by $D_g/GI_g/K/0$. The service time t of a group will be rounded down to w, $w = 0,1,2, \ldots$, if the service of the group who came at time n is finished by the time $n+w+1$ but not before the time $n+w$ where a service time not skipping one time is considered as a base, t is the delay and equals excess over one time unit. Let $P_0; P_{1,}; P_2, \ldots; P_T$ the service time distribution of a group of customers and the state of the system at time $n-0$ can be characterized by a K- duple of integers where a standard argument of the queuing theory proofs that the system with these states clarifys a time-homogeneous, finite or countable Markov chain.

The second loss system is constructed as follows:

Consider the service system $GI_s/GI_s/1/0$ with a group of r customers, where a group arrives each time unit and the service time of the customers equals t time units, while both the distributions of the arrivals and the service time of the customers are unspecified. The total number of servers equals 1 only, where all the r customers will be served by this server. That is we have r service times for r customers of the group are served by the same server. 0 means that there are no waiting places. Such a service system is a loss system with r multiservice customers and will be denoted by $D_s/GI_s/1/0$, where a customer arrives each time unit. The service time t will be rounded down to 0, if the service of a customer, who came at time n, is finished at time $n+1-0$ (that is before time $n+1$); rounded down to 1, if the service is finished by at time $n+2-0$ but not before $n+1$, etc.. In general, the service time t will be rounded down to T, if the service of a customer (who came at time n) is finished at time $n+1+T-0$, where the service time of a customer cannot exceed T time units.

3. Service time of a group of r customers and its distribution

To find the service time of a group of r customers, we prove the following theorem.

Theorem he service time of a group of r customers equals the total service times of r customers of the group.

Proof The compound service times of r customers will be rounded down to r (with time delay equals $r-1$), the service of a customer who came at time $n+j-1$ is finished by the time $n+j$, where the j^{th} customer is served without delay; for all $j=1,2,\dots,r$. In this case, the service of a group of r customers who started at time n is finished by the time $n+r$, and the service time of the group will be rounded down to r where it equals to the compound service times of the r customers. The compound service times of r customers will be rounded down to $r+1$(i.e, the r customers are delayed r time units), if the j^{th} customer is served without delay by the time $n+j$, for all $j=1,2,\dots,r-i$; $i=1,2,\dots,r$, and the $(r-i+1)^{th}$ customer is served with time delay (equivalent a time unit) and finishes its service by the time $n+(j+1)+1$. The customers coming during the service of the $(r-i+1)^{th}$ delayed customer will be lost. Consequently, the service of a group of r customers is finished by the time $n+r+1$, and the service time of the group will be rounded down to $r+1$ where it equals the compound service times of at most r customers. Here at most r customers will be served for, the service with delay of the $(r-i+1)^{th}$ customer makes the next coming customers are lost, where the number of lost customers may be $1,2,\dots,r-1$.

In general, the compound service times of r customers will be rounded down to t, where $t=1,2,\dots,T$, if the j^{th} customer is served without delay by the time $n+j$; for all $j=1,2,\dots,r-i$; $i=1,2,\dots,r$, and the $(r-i+1)^{th}$ customer is served with time delay (equals t time units) and finishes its service by the time $n+(j+1)+t$. Therefore, the service of a group of r customers is finished by the time $n+r+t$, and the service time of the group will be rounded down $r+t$ where it equals the compound service times of at most r customers. This completes the proof. The compound service time distribution of the two loss systems is considered in the following way.

Denote by $p_0;p_1;p_2;\dots;p_T$, the distribution of the service time of a customer. Let $c_i(t)$ be the event that the j^{th} customer is served without delay with probability p_0, for all $j=1,2,\dots,r-i$; $i=1,2,\dots,r$, and the $(r-i+1)^{th}$ customer is is served with time delay $t=1,2,\dots T$ time units, with probability p_t. Since the service times of the r customers are independent random variables; it can be seen that the service time distribution of the group eq $P\left(\cap_{i=1}^{r} c_i(t)\right)$

$$\begin{aligned} &= p_0^{r-i}p_t\,, \quad i=1,2,\dots,r\,; \quad t=1,2,\dots,T \\ &= p_0^{r} \qquad, \quad i=1 \qquad ; \quad t=0 \end{aligned} \tag{1}$$

4. Description of the working mechanism of the proposed systems

Consider the service system with two servers $K=2$ in the case that a group of two customers $r=2$ arrives to the system each time unit $i=1$. Assume that each customer arrives to the server each time unit with service time distributions p_0,p_1 where the customer is served without delay or with maximum time delay $(T=1)$.

In this case, the service time distributions of the group is P_1,P_2 where P_1 can be expressed as follow: each customer is served without delay with probability p_0 or the first is served after one time unit with probability p_1 and the second is lost. On the other hand P_2 can be expressed as the first customer is served without delay and the second is delayed one time unite. That is,

$$P_1=p_0^2+p_1, P_2=p_0p_1 \tag{2}$$

If $r=3$, we have

$$P_1=p_1, P_2=p_0^3+p_0p_1,\ P_3=p_0^2p_1 \tag{3}$$

In case $r=4$, we have

$$P_1=p_1,\ P_2=p_0p_1,\ P_3=p_0^4+p_0^2p_1,\ P_4=p_0^3p_1, \tag{4}$$

For all $r=2,3,\dots..$, the group of equations (2), (3), (4) , we can deduce

$$\left.\begin{aligned} P_r &= \textstyle\sum_{v=0}^{r-1} p_0^{r-1-v} p_{v+1}, \\ P_{r-1} &= \textstyle\sum_{v=0}^{r-1} p_0^{r-1-v} p_v, \\ P_{r-2} &= \textstyle\sum_{v=0}^{r-1} p_0^{r-1-v} p_{v-1}, \end{aligned}\right\} \quad (5)$$

where the probability of the smallest service time of the group of r customers will be $P_1 = \sum_{v=0}^{r-1} p_0^{r-1-v} p_{v-r+2}$, $p_0^x p_y = p_0^{x+1} = 0$, $x < r-1$, $p_s = 0$ for $s > 1$, $P_0 = 0$

If $T = 2$, in this cas the service time distribution of the customer is p_0, p_1, p_2 and the service time distribution of the group can be derived as follows:

For $r = 2$, we have

$$P_1 = p_0^2 + p_1, P_2 = p_0 p_1 + p_2,\ P_3 = p_0 p_2 \quad (6)$$

For $r = 3$, we have

$$P_1 = p_1,\ \ P_2 = p_0^3 + p_0 p_1 + p_2, P_3 = p_1 + p_0 p_2,\ \ P_4 = p_0^2 p_2 \quad (7)$$

For $r = 4$, we have

$$P_1 = p_1,\ P_2 = p_0 p_1 + p_2,\ P_3 = p_0^4 + p_0^2 p_1 + p_0 p_2, P_4 = p_0^3 p_1 + p_0^2 p_2,\ P_5 = p_0^3 p_2 \quad (8)$$

we can deduce that the service time distribution of a group of r customers , $r = 2,3, \ldots.,$

$$\left.\begin{aligned} P_{r+1} &= \textstyle\sum_{v=0}^{r-1} p_0^{r-1-v} p_{v+2}, \\ P_r &= \textstyle\sum_{v=0}^{r-1} p_0^{r-1-v} p_{v+1}, \\ P_{r-1} &= \textstyle\sum_{v=0}^{r-1} p_0^{r-1-v} p_v, \\ P_{r-2} &= \textstyle\sum_{v=0}^{r-1} p_0^{r-1-v} p_{v-1}, \\ P_{r-3} &= \textstyle\sum_{v=0}^{r-1} p_0^{r-1-v} p_{v-2}, \end{aligned}\right\} \quad (9)$$

$$P_1 = \sum_{v=0}^{r-1} p_0^{r-1-v} p_{v-r+2} \qquad p_0^x p_y = p_0^{x+1} = 0,$$
$$x < r-1,\ \ p_s = 0\ \ for\ \ s > 2,\ \ P_0 = 0$$

If $T = 3$, then the service time distribution of the customer is p_0, p_1, p_2, p_3

For $r = 2$, we get

$$P_1 = p_0^2 + p_1, P_2 = p_2 + p_0 p_1,$$
$$P_3 = p_0 p_2 + p_3, P_4 = p_0 p_3 \quad (10)$$

For $r = 3$, we get

$$P_1 = p_1,\ P_2 = p_0^3 + p_0 p_1 + p_2,$$
$$P_3 = p_0^2 p_1 + p_0 p_2 + p_3,\ P_4 = p_0^2 p_2 p_0 p_3,$$
$$P_5 = p_0^2 p_3 \quad (11)$$

For $r = 4$, we get

$$P_1 = p_1,\ P_2 = p_0 p_1 + p_2,\ P_3 = p_0^4 + p_0^2 p_1 + p_0 p_2 + p_3,$$
$$P_4 = p_0^3 p_1 + p_0^2 p_2 + p_0 p_3,$$
$$P_5 = p_0^3 p_2 + p_0^2 p_3,\ P_6 = p_0^3 p_3 \quad (12)$$

The service time distributions of a group of r customers, $= 2, 3, \ldots\ldots$, can be derived as

$$\left.\begin{aligned} P_{r+2} &= \textstyle\sum_{v=0}^{r-1} p_0^{r-1-v} p_{v+3}, \\ P_{r+1} &= \textstyle\sum_{v=0}^{r-1} p_0^{r-1-v} p_{v+2}, \\ P_r &= \textstyle\sum_{v=0}^{r-1} p_0^{r-1-v} p_{v+1}, \\ P_{r-1} &= \textstyle\sum_{v=0}^{r-1} p_0^{r-1-v} p_v, \end{aligned}\right\} \quad (13)$$

$$P_1 = \sum_{v=0}^{r-1} p_0^{r-1-v} p_{v-r+2} \qquad p_0^x p_y = p_0^{x+1} = 0,$$
$$x < r-1, \qquad p_s = 0\ \ for\ \ s > 3,$$
$$P_0 = 0$$

In general, for all $T = 1,2,3, \ldots.$, and by the group of equations (5), (9) (13) , we deduce that the service time distribution of a group of r customers,$r = 2, 3, \ldots.$,can be obtained by the system

$$P_{r+T-j} = \sum_{v=0}^{r-1} p_0^{r-1-v} p_{T-j+1+v}$$
$$j = 1, 2, \ldots., r+T-1, r = 2,3, \ldots.., \quad (14)$$
$$p_0^x p_y = p_0^{x+1} = 0, \qquad x < r-1,\ \ p_s = 0\ \ for\ \ s > T,$$
$$P_0 = 0$$

Now, we show that (14) is true.

First, we show that (14) is true in case $T = 1$, that is, we show that

$$P_{r+1-j} = \textstyle\sum_{v=0}^{r-1} p_0^{r-1-v} p_{v-j+2}$$
$$j = 1, 2, \ldots., r \quad (15)$$

Since, for $j = 1, 2, \ldots., r$, (15) gives the group of equations (5). This shows that (15) is true

Secondly, substitute $T = 2$ in (14), we get

$$P_{r+2-j} = \textstyle\sum_{v=0}^{r-1} p_0^{r-1-v} p_{v-j+3}$$
$$j = 1, 2, \ldots., r+1 \quad (16)$$

For $j = 1, 2, \ldots., r+1$, (16) gives the group of equations (9). This shows that (16) is true

Assume that (14) is true for $T = u$, that is :

$$P_{r+u-j} = \textstyle\sum_{v=0}^{r-1} p_0^{r-1-v} p_{u-j+1+v}$$
$$j = 1, 2, \ldots., r+u-1, r = 2,3, \ldots..,$$
$$u\ is\ one\ value\ only,\ \ p_s = 0\ \ for$$
$$s > u, p_0^x p_y = p_0^{x+1} = 0, \qquad x < r-1, \qquad P_0 = 0 \quad (17)$$

is true by assumption.

We prove that (14) is true for T=u+1, that is, we show that

$$P_{r+u+1-j} = \textstyle\sum_{v=0}^{r-1} p_0^{r-1-v} p_{u+1-j+1+v} \qquad j = 1, 2, \ldots., r+u, r = 2,3, \ldots.., \quad (18)$$

$u\ is\ one\ value\ only, p_s = 0\ \ for\ \ s > u+1, p_0^x p_y = p_0^{x+1} = 0, x < r-1, P_0 = 0$, is true

For $j' = j-1 = 1, 2, \ldots., r+u-1$, then, (18) becomes
$P_{r+u-j'} = \sum_{v=0}^{r-1} p_0^{r-1-v} p_{u-j'+1+v}$
$j' = 1, 2, \ldots., r+u-1, r = 2,3, \ldots.,$ (19)
$u \; is \; one \; value \; only, \;\; p_s = 0 \;\; for \;\; s > u, p_0^x p_y = p_0^{x+1} = 0, x < r-1, P_0 = 0$

The group of equations (18) is true for it is the same as (or it is corresponds) the system (17). This display that the group of equations (18) is true for $j = 2, \ldots., r+u$.

Now we show that the system of equations (18) is true for $j = 1$. That is we show (20) is true where:

$$P_{r+u} = p_0^{r-1} p_{u+1} + p_0^{r-2} p_{u+2} + \cdots + p_{u+r}, r = 2,3, \ldots, p_0^x p_y = p_0^{x+1} = 0, x < r-1, \quad (20)$$

The left hand side of the system (20) equals the probability that a group who arrives at time n and served at time $n+r+u+1-0$, u is the time delay of the customer. The probability P_{r+u} can be expressed by different services of its r customers, where each one arrives to the server each time unit, such as each one of the $(r-1)$ customers is served without delay with probability p_0 and the r^{th} customer is served after time delay $u+1$ time units with probability p_{u+1}. The $(r-1)^{th}$ customer completes its service at time $n+r-1-0$ and the r^{th} customer starts its service at time $n+r-1$ and completes its service at time $(n+r-1)+(u+1)+1-0 = n+r+u+1-0$. Therefore, the group who arrives at time n completes its service at time $n+r+u+1-0$ with probability $p_0^{r-1} p_{u+1}$. Note that the first $(r-1)$ customers complete their services at time $n+r-1-0$, with probability p_0^{r-1} without delay. Another option for the expression of the probability P_{r+u} is that the first $(r-2)$ customers complete their services without delay at time $n+r-2-0$ with probability p_0^{r-2} and the $(r-1)^{th}$ customer is served with probability p_{u+2} where it starts its service at time $n+r-2$ and complete its service at time $(n+r-2)+(u+2)+1-0 = n+r+u+1-0$. The r^{th} customer will be lost because there is no waiting place. Therefore, the group of r customers completes its service at time $n+r+u+1-0$ with probability $p_0^{r-2} p_{u+2}$. In general, there exists $g, g = 1,2, \ldots\ldots, r$, options for the expression of the probability P_{r+u} is that the first $(r-g)$ customers complete its service without delay at time $n+r-g-0$ with probability p_0^{r-g} and the $(r-g+1)^{th}$ customer is served with probability p_{u+g} where it starts its service at time $n+r-g$ and complete its service at time $(n+r-g)+(u+g)+1-0 = n+r+u+1-0$.

The $(g-1)$ customers will be lost that is because there is no waiting place. Therefore, the group of r customers completes its service at time $n+r+u+1-0$ with probability $p_0^{r-g} p_{u+g}$. This proofs that the system of equations (20), is true.

Note that, if a customer is delayed, then the next arriving customers will be lost.

5. The service time distribution of a group of r customers

The compound service time distributions of groups of r customers are obtained in the cases that the j^{th} customer is served without delay with probability p_0 for all $j = 1,2, \ldots, r-i; i = 1,2, \ldots, r; r = 2,3, \ldots$; and the $(r-i+1)^{th}$ customer is served with time delay t; $t = 1,2, \ldots, T$ time units, and with probability p_t. Assume that the compound service time of a group is to be found in the case that the first customer is delayed t time units where $t = 1,2, \ldots, T$. Since server is busy, then the next coming x^{th} customer for $x = 2,3, \ldots, r$ will be lost.

In this case, no custumers are served without delay, and $r = i$. Since the service times of the r customers are independent random variables, it can be seen that the compound service time distribution of a group equals P; $P = p_t$ for all $t = 1,2, \ldots, T; r = 2,3, \ldots$.In this case, the compound service time distribution of the delayed groups under the assumption that the first customer is the only served one and the remaining $(r-1)$ customers are lost, is equal to P, where

$$P = p_t, \quad t = 1,2, \ldots, T,$$
$$= 0, \quad t = 0. \quad (21)$$

The $(r-1)$ lost customers are not allowed to be served by another one of the parallel servers of the same loss system even they were empty, for this will affect the number of served arrivals of different coming groups. Also the efficiency of the modified stochastic approximation procedure with delayed groups will be affected. As a result of this case, each group will become a group of one delayed customer and by the last theorem, the service time of each group equals the service time of its delayed customer. Therefore, the service time distribution P of the group equals the service time distribution p_t of its delayed customer. We conclude that the number of served customers of a group will minimizes the number of waste customers of the group to become zero where this leads to raise the performance (efficiency) of the proposed procedure.

6. Technical tools

1. Studying mathematical models of two special service systems.

2. As the considered systems can be viewed as compound Markov chains, the theory of finite or countable (discrete-time) compound Markov chains that were benefitted.

3. The compound states of the chain ready from the elementary zero state was found.

4. The equations substantial for the stationary distribution were found.

5. The set S of compound states that can be arrived from the initial 00…0 under the assumption

$P_i > 0$, for all $0 \leq i \leq T$,

jointly with the identical matrix of transition probabilities P wil be found. The set S is called the basal compound Markov chain.

6. We can easily see that, still assuming $P_i > 0$, for all $0 \leq i \leq T$, the basic compound Markov chain is irreducible, ergodic. Hence, there is a unique stationary distribution π determined by

$$P^T\pi = \pi, \quad T \text{ denotes transpose} \tag{22}$$

7. Some matrix algebra and procedures of solving systems of linear equations were utilized.

8. Asymptotic efficiencies of the investigated stochastic approximated procedure were calculated for different numbers of parallel servers. If $T < K$, then each state α contains at least one 0 and in this case no group of customers will be lost. For $T \geq K$, the unknowns π_α with α containing no $\mathbf{0}$'s can be eliminated from the system (23), as successive transitions from these states to states containing $\mathbf{0}$'s occur deterministically, with probability 1. One of the resting equations can always be deleted as unnecessary; another one is to be joined, namely the equation

$$\sum\nolimits_{\beta} \pi_\beta = 1$$

Solving the reduced system of equations using Matlab Program, and summing the coordinates of the solution, we get the loss probability l, where

$l = \sum_\gamma \pi_\gamma$; γ containing no 0, or complementarily, the efficiency e of the two service systems, where

$$e = \sum\nolimits_\alpha \pi_\alpha; \quad \alpha \text{ contains at least one } 0, \quad l + e = 1. \tag{23}$$

9. Investigating the adjusted stochastic approximation procedure with delayed groups of delayed customers and with allocation of the groups into parallel series.

10.The outcomes obtained for the above-mentioned service systems became main tool of the consideration.

11.Outcomes on almost sure convergence and on asymptotic normality, known for the stochastic approximation procedure without delayed, benefitted.

7. Applications

Two applications will be given here to show that the investigated procedure can be utilized to other stochastic approximation procedure.

Application 1

Consider the case $= 2; T = 3; r = 2$, and assume that $p_0 = 0.9$, $p_1 = p_2 = p_3$; where

$$p_i = \frac{(1 - p_{0)})}{3} = 0.033, \qquad i = 1,2,3$$

Consider the case $= 2; T = 3; r = 2$, and assume that $p_0 = 0.9$, $p_1 = p_2 = p_3$; where

$p_i = (1 - p_{0)})/3 = 0.033, \qquad i = 1,2,3$

Substitute the service time distribution of the customer (p_0, p_1, p_2, p_3) into (10) we get the service time distribution of the group of two customers as $(P_0, P_1, P_2, P_3, P_4)$, where

$P_0 = 0;\ P_1 = 0.843;\ P_2 = P_3 = 0.063;$
$P_4 = 0.03$

Substitute the service time distribution $(P_0, P_1, P_2, P_3, P_4)$ into (22) and solve the resulting system of equations with respect to π; insert the stationary distribution π into (23) to get the efficiency e of the investigated stochastic procedure as $e = 0.85$.

Applications 2

Consider the case $= 2; T = 1; r = 4$, and assume that $p_0 = 0.3$; where $p_1 = 1 - p_0$. Substitute the service time distribution of the customer as (p_0, p_1) into (4) we the service time distribution of the group of four customers as $(P_0, P_1, P_2, P_3, P_4)$, where

$$P_0 = 0;\ P_1 = 0.7;\ P_2 = 0.21;\ P_3 = 0.0711;\ P_4 = .0189;\ E(t) = 1.4089. \tag{24}$$

Substitute the service time distribution $(P_0, P_1, P_2, P_3, P_4)$ into (22) and solve the resulting system of equations with respect to π; insert the stationary distribution π into (23) to get the efficiency e of the investigated stochastic procedure as $e = 0.7848$. The results obtained here show that the modified Robbins-Monro stochastic approximation procedure can serve as a model of stochastic approximation with delayed groups of r customers with efficiency e. That is, the new approach can be investigated by modifying the Robbins-Monro stochastic approximation procedure to be applicable in the presence of the described two loss systems with groups of delayed multiservice r customers.

8. Results and conclusion

Table 1 gives the efficiency e of the investigated stochastic approximation procedure with delayed groups of delayed multiservice r ($r = 2$) customers, where the maximum delay T of a customer is assumed to be 3 the number of servers K equals 2. The efficiencies are calculated for different assumed values of service time distribution of the 2 customers.

Table 2 gives the efficiency e of the investigated stochastic approximation procedure with delayed groups of delayed multiservice r ($r = 4$) customers, where the maximum delay T of a customer is assumed to be 1 and the number of servers K equals 2. The efficiencies are calculated for different assumed values of service time distribution of the 2 customers.

The results obtained show that the modified Robbins-Monro stochastic approximation procedure with delayed observations and with allocation of experiments into K series, can serve as a model of stochastic approximation of delayed groups of r customers, where: observations in the adjusted procedure are groups of delayed customers in the investigated procedure; the delay of an observation in the modified procedure is the service time of a group of r customers; the K series of experiments are the K servers of groups of delayed customers; and the non-existence of a waiting room is the impossibility to accept a group of delayed customers if all the K series are occupied.

Therefore, and as a new result, our approach can be investigated by modifying the Robbins-Monro stochastic approximation procedure to be usable in the existence of the described two loss systems with groups of delayed multiservice r customers.

Comparing the results in Tables 1 and 2, we have the maximum service time of a group ($= r + T - 1$) equals 4 time units while the maximum service time of a customer is 3 and 1, respectively. The efficiencies in Table 1 are less than those in correspondence in Table 2 for $p_0 = 0.1, \dots, 0.5$, where, analytically, the reason is that the maximum service time of a customer in Table 1 is greater than those in correspondence in Table 2. The efficiencies in Table 1 are greater than those in correspondence in Table 2 for $p_0 = 0.6, \dots, 0.9$, where analytically the reason is that the number of customers in Table 1 is less than those in correspondence in Table 2.

The efficiencies in Table 1 are increasing for all $p_0 = 0.1, \dots, 0.9$, and this is expected for the service time distribution p_0 of customers is increasing. In Table 2 the efficiencies are improved and increased for $p_0; p_1 = 0.1, \dots, 0.9$, where this is unexpected for the service time distribution of customers is increasing.

Analytically, efficiencies are decreasing for all increasing service time distribution p_0 of customers and are increasing for all increasing service time distribution p_1 of customers. Also, efficiencies are increased by decreasing the number of customers, or decreasing the service time of customers.

New results are obtained through the presented applications where the efficiencies are improved and increased by increasing the service time distribution of customers as well as by increasing the service times of customers. This situation allows to serve a larger number of customers during the increased service time of customers. As a result of this situation, the efficiencies of the introduced procedure can be improved and increased by increasing the service time of customers where the number of served customers will be increased during their increased service time. According to this new situation, our investigated procedure can be applied into other experiments. Analysis of the two applications shows that the efficiency of the procedure depends on the maximum service time of the customer and on the number of customers during the increased service time of customers during the increased service time of customers. As a result of this situation, the efficiencies of the introduced procedure can be improved and increased by increasing the service time of customers where the number of served customers will be increased during their increased service time. According to this new situation, our investigated procedure can be applied into other experiments. Analysis of the two applications shows that the efficiency of the procedure depends on the maximum service time of the customer and on the number of customers of the group. The results of approximation of the procedures in Tables 1 and 2 seem to be acceptable.

In general, our proposal can be applied to other stochastic approximation procedures to increase the production in many fields such as medicine, computer sciences, industry, and applied sciences.

Table 1. Exact percentage efficiency e of the investigated procedure with delayed groups of r customers ($r = 2$), maximum service time $T(T = 3)$, and its application to the modified Robbins-Monro procedure.

p_0	P_1	P_2	P_3	P_4	$K = 2$ e	$K = 3$ e
0.1	0.31	0.33	0.33	0.030	61.9	85.82
0.2	0.307	0.32	0.32	0.053	61.2	84.91
0.3	0.323	0.303	0.303	0.070	61.2	84.76
0.4	0.36	0.28	0.28	0.080	61.9	85.35
0.5	0.417	0.251	0.251	0.080	63.7	86.69
0.6	0.493	0.213	0.213	0.080	66.2	88.8
0.7	0.59	0.17	0.17	0.070	70.1	91.65
0.8	0.707	0.121	0.121	0.054	76.3	95.03
0.9	0.843	0. 063	0. 063	0.030	85.01	98.33

Note that: $p_1 = p_2 = p_3 = (1 - p_0)/3,\ P_2 = P_3, \quad P_4 = 1 - \sum_{i=1}^{3} P_i$

Table 2. Exact percentage efficiency e of the investigated procedure with delayed group of customers ($r = 4$), maximum service time T ($T = 1$), and its application to the modified Robbins-Monro procedure.

p_0	P_1	P_2	P_3	P_4	$K = 2$ e	$K = 3$ e
0.1	0.9	0.09	0.0091	0.0009	91.47	99.89
0.2	0.8	0.16	0.0336	0.0064	84.6	99.14
0.3	0.7	0.21	0.0711	0.0189	78.48	97.28
0.4	0.6	0.24	0.1216	0.0384	72.8	94.20
0.5	0.5	0.25	0.1875	0.0625	67.51	90.22
0.6	0.4	0.24	0.2736	0.0864	62.65	85.82
0.7	0.3	0.21	0.3871	0.1029	58.34	81.51
0.8	0.2	0.16	0.5376	0.1024	54.69	77.83
0.9	0.1	0. 09	0. 7371	0.0729	51.84	75.356

Note that: $p_1 = 1 - p_0, \quad P_4 = 1 - \sum_{i=1}^{3} P_i$

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